Final Exam: Complex Analysis

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Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two hours to complete this test. Good luck!*

注意事项：卷面分5分,试题总分95分. 其中卷面整洁,书写规范(5分);卷面较整洁,书写较规范(3分); 书写潦草，乱涂乱画(0分).

1.**(10 points)** Suppose the function $f:\mathbb{D}\to\mathbb{C}$ is holomorphic. Show that $2|f'(0)|\le d$, where $d=\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|$.

2.**(40 points**) (1), Evaluate the following integral

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{(z+2)^2}{z^2(2z-1)}\mathrm{d}z.$$

(2), Evaluate the integral

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}dx.$$

(3), Find the number of zeros, counting multiplicities, of the polynomial $f(z)=2z^5+4z^2+1$ in the unit disc $\mathbb{D}$.

(4), Find a conformal map from $\{z\in\mathbb{D}:\Re z>0\}$ to the unit disc $\mathbb{D}$.

(5), Find the Hadamard products for the hyperbolic sine function

$$f(z)=\sinh\pi z=\frac{e^{\pi z}-e^{-\pi z}}{2}.$$

3.**(15 points)** Let $f(z)$ be holomorphic in $\mathbb{D}$ and $|f(z)|\le 1$ for all $z\in\mathbb{D}$. Prove that

$$\frac{|f(0)|-|z|}{1+|f(0)||z|}\le|f(z)|\le\frac{|f(0)|+|z|}{1-|f(0)||z|},\quad z\in\mathbb{D}.$$

4.**(15 points)** Show that the group of automorphisms of $\mathbb{C}$

$$\mathrm{Aut}(\mathbb{C})=\{az+b:\ a,b\in\mathbb{C},a\neq 0\}.$$

5.**(15 points)** (1), Show that the function $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ has an analytic continuation to all of $\mathbb{C}$ as a meromorphic function with simple poles at $s=0$ and $s=1$ and has the functional equation $\xi(s)=\xi(1-s)$, for all $s\in\mathbb{C}$. (Hint: using the following relation $\sum_{n\in\mathbb{Z}}e^{-\pi n^2 t}=t^{-1/2}\sum_{n\in\mathbb{Z}}e^{-\pi n^2/t},\ t>0.$)

(2), Compute the values of $\zeta(0)$ and $\mathrm{Res}_{s=1}\zeta(s)$.