Graduate Algebra Final Exam

School of Mathematics, Shandong University

Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two and a half hours to complete this test. Good luck!*

9:00 - 11:30; July 4, 2018. G.H. JI

注意事项：卷面分5分,试题总分95分. 其中卷面整洁,书写规范(5分);卷面较整洁,书写较规范(3分); 书写潦草，乱涂乱画(0分).

***1.*(10 points)**

(1). Let a and b belong to the group G. If $ab = ba$ and $|a| = m$, $|b| = n$, where $(m, n) = 1$. Show that $|ab| = mn$.

(2). Let G be a finite abelian group and $m = \max\{|a| : a \in G\}$. Then $|b| \big| m$, for any $b \in G$.

***2.*(30 points)**

(1). The multiplicative group of a finite field is cyclic.

(2). Find all the monic irreducible polynomials of degree 2 in $\mathbb{F}_3[x]$.

(3). Construct a finite field with 9 elements and give its the multiplication table.

***3.*(10 points)**

Let G be an abelian group. Then any irreducible representation ρ of G over finite-dimensional complex vector space V has degree one.

***4.*(15 points)**

(1). Show that for any $X, Y \in M_n(\mathbb{C})$, small $t \in \mathbb{R}$,

$$\exp(tX)\exp(tY) = \exp\Big(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\Big).$$

(2). Show that

$$\lim_{k\to\infty}\Big(\exp\big(\frac{X}{k}\big)\exp\big(\frac{Y}{k}\big)\Big)^k = \exp(X+Y).$$

***5.*(15 points)**

Show that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

***6.*(15 points)**

(1). Let K be a splitting field of the polynomial $f(x) = x^3 - 3x + 1$ over $\mathbb{Q}$. Find the degree $[K : \mathbb{Q}]$.

(2). Find the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.