EXPONENTIAL SUMS, PRIME-DETECTING SIEVES, AND THEIR APPLICATIONS IN ADDITIVE PRIME NUMBER THEORY

ANGEL V. KUMCHEV
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In these lectures, we shall describe Harman's "alternative sieve" and some of the twists in its applications to additive problems involving primes. We start with an example from Diophantine approximation, which illustrates the basic ideas behind Harman's sieve in a relatively simple setting. We then discuss the variations and adjustments needed when the method is applied in the typical setting for additive problems involving primes.

LECTURE 1: THE DISTRIBUTION OF αp MODULO ONE. VAUGHAN'S IDENTITY

We want to address the following question.

Question. *Suppose that α is an irrational real number. For what $\theta > 0$ does the Diophantine inequality*

$$\|\alpha p\| < p^{-\theta} \tag{1}$$

have infinitely many solutions with p prime?

Let X be a large real number and Φ be a smooth function supported in $[-X^{-\theta}, X^{-\theta}]$ and such that

$$0 \le \Phi(x) \le 1, \quad \int_{\mathbb{R}} \Phi(x)\, dx = X^{-\theta}.$$

Then $\Psi(x) = \Phi(\|x\|)$ is 1-periodic and has a Fourier expansion of the form

$$\Psi(x) = X^{-\theta} + \sum_{h \ne 0} c_h e(hx),$$

with Fourier coefficients satisfying

$$|c_h| \ll_k \frac{X^{-\theta}}{\big(1 + X^{-\theta}|h|\big)^k} \quad (k \ge 1). \tag{2}$$

We now use Ψ and its Fourier expansion to count the solutions of (1) with $X/2 < p \le X$. Let $S_\alpha(X)$ be the number of such p. Then

$$S_\alpha(X) \ge \sum_{p \sim X} \Psi(\alpha p) = \sum_{p \sim X} \Big(X^{-\theta} + \sum_{h \ne 0} c_h e(\alpha h p) \Big),$$

where we write $p \sim X$ for the condition $X/2 < p \le X$. Let $H = X^{\theta+\varepsilon}$. By (2) with k sufficiently large, the contribution to the last sum from h with $|h| > H$ is tiny. Hence,

$$S_\alpha(X) \ge \frac{X^{1-\theta}}{2\log X}(1 + o(1)) + \sum_{0<|h|\le H} c_h \sum_{p \sim X} e(\alpha h p). \tag{3}$$

Thus, we have reduced the original problem to the estimation of an exponential sum. At this stage, it suffices to show that

$$\max_{0<|h|\le H} \Big| \sum_{p \sim X} e(\alpha h p) \Big| \ll X^{1-\theta-\varepsilon}, \tag{4}$$

and (3) will yield the estimate

$$S_\alpha(X) \ge \frac{X^{1-\theta}}{2\log X}(1 + o(1)).$$

In fact, with a little more care, we can even turn this lower bound into an asymptotic formula of the form

$$S_\alpha(X) = \frac{X^{1-\theta}}{\log X}(1+o(1)).$$

Next, we turn attention to the exponential sum bound (4). How does one prove a bound like that? In 1937, Vinogradov found that if one can estimate two types of double exponential sums, then one can also estimate an exponential sum over primes such as that in (4). That is how he proved the Goldbach–Vinogradov theorem on sums of three primes. To estimate the left side of (4), Vinogradov would consider double sums

$$\sum_{m\sim M}\sum_{mn\sim X} a_m b_n e(\alpha hmn)$$

of one of the following two types:

- Type I: $|a_m| \ll 1$, $b_n = 1$, and M is not "too large";
- Type II: $|a_m| \ll 1$, $|b_n| \ll 1$, and M is neither "too small", nor "too large".

If Vinogradov was able to show that each of these double sums is $\ll X^{1-\theta-2\varepsilon}$, then he could use a rather technical combinatorial technique that he developed to deduce (4). Vinogradov's combinatorial technique was far from transparent, and so for 40 years the estimation of exponential sums over primes remained the domain of a few "specialists".

In 1977, Vaughan discovered an identity which can be used instead of Vinogradov's combinatorial method to pass from estimates for Type I and Type II sums to estimates for sums over primes. Suppose that U is a parameter to be chosen later with $U \le X^{1/2-\varepsilon}$. Then the simplest form of Vaughan's identity gives

$$\begin{aligned}\sum_{k\sim X}\Lambda(k)e(\alpha hk) &= \sum_{m\le U}\sum_{mn\sim X}\mu(m)(\log n)e(\alpha hmn)\\ &\quad - \sum_{r,s\le U}\sum_{rst\sim X}\Lambda(r)\mu(s)e(\alpha hrst)\\ &\quad - \sum_{r,s>U}\sum_{rst\sim X}\Lambda(r)\mu(s)e(\alpha hrst).\end{aligned}$$

If we put

$$a_m = \sum_{\substack{rs=m\\ r,s\le U}}\Lambda(r)\mu(s), \quad b_n = \sum_{\substack{rt=n\\ r>U}}\Lambda(r),$$

we can rewrite the above identity as

$$\begin{aligned}\sum_{k\sim X}\Lambda(k)e(\alpha hk) &= \sum_{m\le U}\sum_{mn\sim X}\mu(m)(\log n)e(\alpha hmn)\\ &\quad - \sum_{m\le U^2}\sum_{mn\sim X}a_m e(\alpha hmn)\\ &\quad - \sum_{m,n>U}\sum_{mn\sim X}\mu(m)b_n e(\alpha hmn).\end{aligned}$$

Note that $|a_m| \le \log m$ and $|b_n| \le \log n$. Hence, the first two sums on the right side of the last identity can be split into subsums of Type I with $M \le U^2$, while the third sum can be split into subsums of Type II with $U \le M \le X/U$. The question now is: Can we choose $U \le X^{1/2-\varepsilon}$ so that we can estimate all those Type I and Type II sums?

Leaving the technical details out, we can report that when α is irrational and the large parameter X is chosen "suitably", we have

$$\sum_{m\sim M}\sum_{mn\sim X} a_m b_n e(\alpha hmn) \ll X^{1-\theta-2\varepsilon},$$

provided that one of the following holds:

- the sum is of Type I with $M \le X^{1-\theta}$;
- the sum is of Type II with $X^\theta \le M \le X^{1-2\theta}$ or $X^{2\theta} \le M \le X^{1-\theta}$.

Suppose first that $\theta < \frac{1}{4}$. Then the two conditions for M in the Type II sum bound overlap. Hence, we can estimate Type I sums with $M \le X^{1-\theta}$ and Type II sums with $X^{\theta} \le M \le X^{1-\theta}$. Let us compare the constraints

$$M \le X^{1-\theta}, \quad X^{\theta} \le M \le X^{1-\theta} \tag{5}$$

with the ranges

$$M \le U^2, \quad U \le M \le X/U \tag{6}$$

that emerged from the application of Vaughan's identity. We notice that if we choose $U = X^{\theta}$, inequalities (5) imply inequalities (6). Hence, when $\theta < \frac{1}{4}$, we can apply Vaughan's identity with $U = X^{\theta}$ and the above double-sum estimates to prove (4).

How do things change when $\theta > \frac{1}{4}$? Then we do not have an estimate for Type II sums with $X^{1-2\theta} \le M \le X^{2\theta}$, and so there is no choice of U such that we can estimate all Type II sums with $U \le M \le X/U$. Thus, the above strategy fails when $\theta > \frac{1}{4}$. In other words, given the above Type I and Type II sum estimates, we can use Vaughan's identity to prove (4) only when $\theta < \frac{1}{4}$.

Lecture 2: The distribution of αp modulo one. A simple sieve instead of Vaughan's identity

To summarize the state of our knowledge: When $0 < \theta < \frac{1}{4}$, we have an asymptotic formula

$$S_{\alpha}(X) = \frac{X^{1-\theta}}{\log X}(1 + o(1)),$$

whereas when $\theta > \frac{1}{4}$, we have no result. Furthermore, the error term in the asymptotic formula comes in the form

$$\sum_{M \in \mathcal{M}} \Big| \sum_{m \sim M} \sum_{mn \sim X} a_m^{(M)} b_n^{(M)} e(\alpha hmn) \Big|,$$

where the double sums are either Type I or Type II and the set $\mathcal{M}$ of choices for M is "small"—having $O(\log X)$ elements. When we deal with this error term, the difference between $\theta = \frac{1}{4} - \varepsilon$ and $\theta = \frac{1}{4} + \varepsilon$ is that in the former case we can estimate all the double sums that arise, whereas in the latter we can estimate all the double sums except for a small number of Type II sums with $X^{1/2-\varepsilon} \le M \le X^{1/2+\varepsilon}$. Yet, much to our frustration, even though we have lost control over just a few double sums, we have lost the result completely.

Harman's alternative sieve is designed to achieve further progress in situations such as that described above. In this lecture, we demonstrate how to use Harman's sieve to give an alternative proof of the result from Lecture 1. This alternative proof, however, will have the advantage that it makes transparent how one can turn reach beyond $\theta = \frac{1}{4}$ if one replaces the above asymptotic formula with a lower bound.

The proof in Lecture 1 had two parts: harmonic-analytic (Fourier series) and combinatorial (Vaughan's identity), and the harmonic analysis came first. This time we interchange the order of the harmonic and combinatorial analysis. That is, we start with the combinatorial analysis and apply it directly to $\sum_p \Psi(\alpha p)$ (as opposed to the exponential sums that appear in the harmonic analysis of this sum). For $z \ge 2$, we now define the arithmetic function

$$\psi(n,z) = \begin{cases} 1 & \text{if } n \text{ has no prime divisor } p \text{ with } p \le z, \\ 0 & \text{otherwise.} \end{cases}$$

It is also convenient to extend the definition of $\psi(n,z)$ to all positive *real* n by setting $\psi(n,z) = 0$ when $n \notin \mathbb{Z}$. In particular, for integers n with $n \sim X$, $\psi(n, X^{1/2})$ is simply the indicator function of the primes—$\psi(n, X^{1/2})$ is 1 or 0 according as n is prime or composite. Hence,

$$\sum_{p \sim X} \Psi(\alpha p) = \sum_{n \sim X} \psi(n, X^{1/2}) \Psi(\alpha n). \tag{7}$$

The combinatorial argument is based on *Buchstab's identity*,

$$\psi(n, z_2) = \psi(n, z_1) - \sum_{z_1 < p \le z_2} \psi(n/p, p) \qquad (2 \le z_1 < z_2),$$

which is merely a form of the inclusion-exclusion principle. Applying Buchstab's identity to the right side of (7), we get

$$\begin{aligned}\sum_{n\sim X}\psi(n,X^{1/2})\Psi(\alpha n) &= \sum_{n\sim X}\psi(n,z)\Psi(\alpha n) - \sum_{n\sim X}\sum_{z<p\le X^{1/2}}\psi(n/p,p)\Psi(\alpha n)\\ &= \sum_{n\sim X}\psi(n,z)\Psi(\alpha n) - \sum_{z<p\le X^{1/2}}\sum_{mp\sim X}\psi(m,p)\Psi(\alpha mp)\\ &= \Sigma_1-\Sigma_2.\end{aligned}$$

Let us now take a look at the double sum Σ_2 in the above decomposition. Using the Fourier expansion of Ψ and the bound (2) for its Fourier coefficients, we obtain

$$\begin{aligned}\Sigma_2 &= \sum_{z<p\le X^{1/2}}\sum_{mp\sim X}\psi(m,p)\Big(X^{-\theta}+\sum_{0<|h|\le H}c_h e(\alpha hmp)\Big)+O(1)\\ &= X^{-\theta}T_2(0)+O\Big(\sum_{0<|h|\le H}\frac{1}{|h|}|T_2(h)|+1\Big),\end{aligned}$$

where

$$T_2(h)=\sum_{z<p\le X^{1/2}}\sum_{mp\sim X}\psi(m,p)e(\alpha hmp).$$

The sum $T_2(0)$ that appears in the main term can be evaluated using standard results and techniques (mainly the PNT and partial summation: see Exercise 2). We have

$$T_2(0)=\frac{X}{2\log X}(c_2+o(1)),\qquad c_2=\int_\zeta^{1/2}\omega\Big(\frac{1-t}{t}\Big)\frac{dt}{t^2}=\int_2^{1/\zeta}\omega(u-1)\,du, \tag{8}$$

where $z=X^\zeta$ and ω is *Buchstab's function*, defined as the continuous solution of the differential delay equation

$$\begin{cases}\omega(u)=1/u & \text{when } 1\le u\le 2,\\ (u\omega(u))'=\omega(u-1) & \text{when } u>2.\end{cases}$$

The superficial technical details aside, the evaluation of the main term is considered "easy" and we are really interested in the sums $T_2(h)$, with $0<|h|\le H$, appearing in the remainder term above. Those sums resemble Type II sums but for one important detail: the coefficients $\psi(m,p)$ are not products of the form $a_m b_p$. Since that structure of the coefficients of the Type II sum plays a central role in its estimation, this difference is an obstacle to the direct estimation of $T_2(h)$. However, by a simple trick using Perron's formula (see Exercise 3), we can show that

$$T_2(h)\ll(\log X)\Big|\sum_{z<p\le X^{1/2}}\sum_{mp\sim X}a_m b_p e(\alpha hmp)\Big|,$$

where the coefficients a_m and b_p are complex numbers with $|a_m|\le 1$ and $|b_p|\le 1$. The latter sum can be split into $O(\log X)$ subsums, each of which is a genuine Type II sum with $z\le M\le X^{1/2}$. Therefore, when $0<\theta<\frac14$ and $z\ge X^\theta$, we can use our Type II sum bound to deduce that

$$\Sigma_2=\frac{X^{1-\theta}}{2\log X}(c_2+o(1)).$$

Is it possible to use the same approach to obtain an approximation for Σ_1? If we were to try, we would discover that we need an asymptotic formula similar to (8) for $T_1(0)$ and an upper bound for $T_1(h)$, with $0<|h|\le H$, where

$$T_1(h)=\sum_{n\sim X}\psi(n,z)e(\alpha hn).$$

Since $T_1(h)$ is not a double sum, it is neither of Type I nor of Type II, and thus, we currently know no bound for it. It turns out, however, that we can derive a bound for this sum (and for more general sums) from bounds for Type I and Type II sums. We shall defer the explanation how this is done and shall state a lemma that can be deduced from the Type I and Type II sum estimates stated above.

Lemma 1. *Let α, X and h be as above. Suppose that $M \le X^{1-\theta}$, $z \le X^{1-3\theta}$, and (a_m) is a complex sequence, with $|a_m| \le 1$. Then*

$$\sum_{m \sim M} \sum_{mn \sim X} a_m \psi(n,z) e(\alpha hmn) \ll X^{1-\theta-\varepsilon}.$$

The kind of sum that appears in this lemma is not quite as general as a Type II sum, but it is more general than a Type I sum. Indeed, if $z = 1$, $\psi(n,z) = 1$ and the above sum turns into a Type I sum. We shall refer to this type of sum as a Type I/II sum. Note that the sum $T_1(h)$ above is a Type I/II sum with $M = 1$ (which is acceptable in the lemma). Thus, if $z \le X^{1-3\theta}$, we can use Lemma 1 to estimate $T_1(h)$ and to show

$$\Sigma_1 = \frac{X^{1-\theta}}{2\log X}(c_1 + o(1)), \qquad c_1 = \zeta^{-1}\omega(1/\zeta).$$

When $0 < \theta < \frac{1}{4}$, we can choose $z = X^{1/4}$ in the above analysis of Σ_1 and Σ_2 to obtain an alternative proof of the asymptotic formula

$$\sum_{p \sim X} \Psi(\alpha p) = \frac{X^{1-\theta}}{2\log X}(c_1 - c_2 + o(1)) = \frac{X^{1-\theta}}{2\log X}(1 + o(1)). \tag{9}$$

Sketch of the proof of Lemma 1. Let $P(z) = \prod_{p \le z} p$. By the properties of the Möbius function, the given exponential sum is

$$\sum_{m \sim M} \sum_{mn \sim X} \sum_{d | (n, P(z))} a_m \mu(d) e(\alpha hmn) \ll (\log X) \Big| \sum_{m \sim M} \sum_{\substack{d | P(z) \\ d \sim D}} \sum_{mkd \sim X} a_m \mu(d) e(\alpha hmkd) \Big|,$$

for some $D \ll X/M$. We consider two cases:

Case 1: $MD \ll X^{1-\theta}$. Then the above sum can be rewritten as a Type I sum.

Case 2: $MD \gg X^{1-\theta}$. Then $d = p_1 p_2 \cdots p_s$, $s \ge 1$, where the p_j's are primes with

$$p_1 < p_2 < \cdots < p_s \le z, \quad mp_1 p_2 \cdots p_s \gg X^{1-\theta}.$$

Let $1 \le t \le s$ be such that $mp_1 p_2 \cdots p_t \gg X^{1-\theta} \gg mp_1 p_2 \cdots p_{t-1}$. Then

$$X^{2\theta} \ll mp_1 p_2 \cdots p_{t-1} \ll X^{1-\theta}.$$

We can use this observation and Exercise 3 to bound the given sum by a linear combination of a $O\big((\log X)^c\big)$ Type II sums. □

We remark that the above proof is essentially Vinogradov's original method for estimation of sums over primes.

Lecture 3: The distribution of αp modulo one. A simple lower-bound sieve

Since we already had a result for $\theta < \frac{1}{4}$, we should ask whether we have gained any insight by recasting the proof in an alternative form. Let us consider again how things change as θ crosses over the $\frac{1}{4}$ threshold. When $\theta > \frac{1}{4}$, we no longer can choose z with $X^{\theta} \le z \le X^{1-3\theta}$, so we set $z = X^{1-3\theta}$. The analysis of Σ_1 then remains the same for $\theta < \frac{1}{3}$. We now write

$$\begin{aligned}\Sigma_2 &= \Big\{ \sum_{z < p < X^{\theta}} + \sum_{X^{\theta} \le p \le X^{1-2\theta}} + \sum_{X^{1-2\theta} < p \le X^{1/2}} \Big\} \sum_{mp \sim X} \psi(m,p)\Psi(\alpha mp) \\ &= \Sigma_3 + \Sigma_4 + \Sigma_5.\end{aligned}$$

We can use our Type II sum bound to evaluate Σ_4 similarly to how we evaluated the entire Σ_2 in the case $\theta < \frac{1}{4}$. This yields

$$\Sigma_4 = \frac{X}{2\log X}(c_4 + o(1)), \qquad c_4 = \int_{\theta}^{1-2\theta} \omega\Big(\frac{1-t}{t}\Big)\frac{dt}{t^2}.$$

We now turn to Σ_3. The harmonic analysis of this sum produces exponential sums of Type II that we cannot estimate. However, we can apply Buchstab's identity to Σ_3 to get

$$\begin{aligned}\Sigma_3 &= \sum_{z<p<X^\theta} \sum_{mp\sim X} \psi(m,z)\Psi(\alpha mp) - \sum_{z<p<X^\theta} \sum_{mp\sim X} \sum_{z<q\le p} \psi(m/q,q)\Psi(\alpha mp) \\ &= \sum_{z<p<X^\theta} \sum_{mp\sim X} \psi(m,z)\Psi(\alpha mp) - \sum_{z<q\le p<X^\theta} \sum_{kpq\sim X} \psi(k,q)\Psi(\alpha kpq) = \Sigma_6 - \Sigma_7;\end{aligned}$$

here q denotes a prime number too. Note that Σ_6 is a Type I/II sum, which can be evaluated similarly to Σ_1; we have

$$\Sigma_6 = \frac{X}{2\log X}(c_6 + o(1)), \qquad c_6 = \frac{1}{1-3\theta}\int_{1-3\theta}^{\theta} \omega\Big(\frac{1-t}{1-3\theta}\Big)\frac{dt}{t}.$$

We can give a similar further decomposition for Σ_5. However, before we do that, we note that in the case of Σ_5, the only integers m with $mp \sim X$ and $\psi(m,p) \ne 0$ are the primes $q \sim X/p$. Hence, on writing $Y_p = (X/p)^{1/2}$, we have

$$\begin{aligned}\Sigma_5 &= \sum_{X^{1-2\theta}<p\le X^{1/2}} \sum_{mp\sim X} \psi(m,Y_p)\Psi(\alpha mp) \\ &= \sum_{X^{1-2\theta}<p\le X^{1/2}} \sum_{mp\sim X} \psi(m,z)\Psi(\alpha mp) - \sum_{\substack{X^{1-2\theta}<p\le X^{1/2} \\ z<q\le Y_p}} \sum_{kpq\sim X} \psi(k,q)\Psi(\alpha kpq) = \Sigma_8 - \Sigma_9.\end{aligned}$$

Again, Σ_8 is a Type I/II sum and we have

$$\Sigma_8 = \frac{X}{2\log X}(c_8 + o(1)), \qquad c_8 = \frac{1}{1-3\theta}\int_{1-2\theta}^{1/2} \omega\Big(\frac{1-t}{1-3\theta}\Big)\frac{dt}{t}.$$

Combining all the decompositions and evaluations, we now obtain

$$\sum_{p\sim X} \Psi(\alpha p) = \frac{X^{1-\theta}}{2\log X}(c_1 - c_4 - c_6 - c_8 + o(1)) + \Sigma_7 + \Sigma_9. \tag{10}$$

We still do not know how to evaluate Σ_7 and Σ_9, but they have one important advantage over their exponential sum counterparts—they are non-negative! Hence, (10) gives

$$\sum_{p\sim X} \Psi(\alpha p) \ge \frac{X^{1-\theta}}{2\log X}(c_1 - c_4 - c_6 - c_8 + o(1)), \tag{11}$$

which is non-trivial whenever $c_1 - c_4 - c_6 - c_8 > 0$. It is easy to see that this already supersedes the earlier result. Indeed, when $\theta = \frac{1}{4} + \varepsilon$, with $\varepsilon > 0$ small, one can show that the sum $c_4 + c_6 + c_8$ is close to c_2, whence $c_1 - c_4 - c_6 - c_8$ is close to 1. Thus, we can get a result with $\theta = \frac{1}{4} + \varepsilon$ for *some* $\varepsilon > 0$. Furthermore, using numerical integration to evaluate the above integrals, we find that when $\theta = 0.28$,

$$c_1 - c_4 - c_6 - c_8 \ge 0.32.$$

Therefore, we conclude that (1) has infinitely many solutions in prime p for $\theta \le 0.28$.

In fact, we can easily do even better. Before we get to that though, let us analyze the constant $c_1 - c_4 - c_6 - c_8$ appearing in the lower bound (11). We decomposed the left side of (11) as

$$\Sigma_1 - \Sigma_4 - \Sigma_6 + \Sigma_7 - \Sigma_8 + \Sigma_9.$$

We expect that

$$\Sigma_i = \frac{X^{1-\theta}}{2\log X}(c_i + o(1))$$

for all six values of i that appear in the above decomposition, but we can prove those asymptotic formulas only for $i = 1, 4, 6$, and 8. For Σ_7 and Σ_9, we *expect* such asymptotic formulas with

$$c_7 = \int_{1-3\theta}^{\theta}\int_{1-3\theta}^{u} \omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2} \quad \text{and} \quad c_9 = \int_{1-2\theta}^{1/2}\int_{1-3\theta}^{(1-u)/2} \omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2},$$

respectively, but we cannot prove them. However, it follows easily from our combinatorial decomposition that

$$c_1 - c_4 - c_6 + c_7 - c_8 + c_9 = 1,$$

whence

$$c_1 - c_4 - c_6 - c_8 = 1 - c_7 - c_9.$$

We note that c_7 and c_9 are the constants in the main terms of the two expected asymptotic formulas that we missed. One may think of this as follows: Each sum (like Σ_7 and Σ_9 above) which we cannot evaluate and estimate trivially results in a "loss" being subtracted from the expected asymptotic formula for the original sum; the resulting lower bound is non-trivial if the total of such losses does not exceed the expected main term (which has a constant coefficient equal to 1). This observation is useful in the final stage of the application of the method where numeric integration is needed.

LECTURE 4: THE DISTRIBUTION OF αp MODULO ONE. A FULL-SCALE LOWER-BOUND SIEVE

In the analysis in Lecture 3, we discarded Σ_7 and Σ_9 entirely, thus incurring losses that were larger than necessary. We now describe the basic tools used to reduce those losses. For example, let us take a closer look at Σ_9. For convenience, we assume that $\frac{1}{4} < \theta < \frac{3}{10}$. Then, we have $Y_p \le X^{1-2\theta}$, and the sum

$$\Sigma_{10} = \sum_{\substack{X^{1-2\theta} < p \le X^{1/2} \\ X^{\theta} \le q \le Y_p}} \sum_{kpq \sim X} \psi(k,q)\Psi(\alpha kpq)$$

is a part of Σ_9 that can be evaluated using our estimates for Type II sums. Hence,

$$\Sigma_{10} = \frac{X^{1-\theta}}{2\log X}(c_{10} + o(1)), \qquad c_{10} = \int_{1-2\theta}^{1/2} \int_{\theta}^{(1-u)/2} \omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2}.$$

Note that c_{10} is a part of the integral defining c_9. Similarly, we can evaluate the part of Σ_9 where

$$z < q < X^{\theta}, \quad X^{2\theta} \le pq \le X^{1-\theta}.$$

Thus,

$$\Sigma_9 \ge \frac{X^{1-\theta}}{2\log X}(c_9 - \ell_9 + o(1)), \tag{12}$$

where ℓ_9 is the constant in the expected asymptotic formula for the unevaluated parts of Σ_9. Hence,

$$\ell_9 = \iint_{\mathcal{D}_9} \omega\Big(\frac{1-u-v}{v}\Big)\frac{du\,dv}{uv^2},$$

where $\mathcal{D}_9$ is the part of the uv-plane subject to the constraints,

$$1 - 2\theta < u < 1/2, \quad 1 - 3\theta < v < \theta, \quad u + v \notin [2\theta, 1-\theta].$$

If we use (12) instead of the trivial bound $\Sigma_9 \ge 0$ to estimate the right side of (10), we can replace (11) by

$$\sum_{p\sim X} \Psi(\alpha p) \ge \frac{X^{1-\theta}}{2\log X}(1 - c_7 - \ell_9 + o(1)),$$

Next, let us take a second look at

$$\Sigma_7 = \sum_{z < p_2 \le p_1 < X^{\theta}} \sum_{kp_1p_2 \sim X} \psi(k, p_2)\Psi(\alpha k p_1 p_2).$$

Here too, we find a subsum that we can evaluate using our Type II sum bound—that is the part of Σ_7 where

$$z < p_2 \le p_1 < X^{\theta}, \quad X^{\theta} \le p_1p_2 \le X^{1-2\theta}.$$

This leaves two parts of Σ_7 which we cannot evaluate and which are subject to the conditions

$$p_1p_2 < X^{\theta} \quad \text{and} \quad p_1p_2 > X^{1-2\theta},$$

respectively. Let Σ_{11} denote the part of Σ_7 subject to $p_1 p_2 > X^{1-2\theta}$. The sum with $p_1 p_2 < X^\theta$ may be empty (it *is* empty when $\theta \le \frac{2}{7}$), but even when this sum is non-empty we can evaluate most of it by further use of Buchstab's identity. Two more appeals to Buchstab's identity yield

$$\begin{aligned}\sum_{\substack{z<p_2\le p_1\\ p_1p_2<X^\theta}} \sum_{mp_1p_2\sim X} \psi(m,p_2)\Psi(\alpha m p_1 p_2) &= \sum_{\substack{z<p_2\le p_1\\ p_1p_2<X^\theta}} \sum_{mp_1p_2\sim X} \psi(m,z)\Psi(\alpha m p_1 p_2)\\ &\quad - \sum_{\substack{z<p_3\le p_2\le p_1\\ p_1p_2<X^\theta}} \sum_{mp_1p_2p_3\sim X} \psi(m,z)\Psi(\alpha m p_1 p_2 p_3)\\ &\quad + \sum_{\substack{z<p_4\le p_3\le p_2\le p_1\\ p_1p_2<X^\theta}} \sum_{mp_1\cdots p_4\sim X} \psi(m,p_4)\Psi(\alpha m p_1 \cdots p_4).\end{aligned}$$

Note that the first two sums on the right can be evaluated using our Type I/II sum bound. Furthermore, we can use a Type II sum estimate to evaluate any part of the quintuple sum over $m, p_1, \dots, p_4$ in which a subproduct of $p_1p_2p_3p_4$ lies between X^θ and $X^{1-2\theta}$. The remaining ("bad") parts of the quintuple sum we shall call Σ_{12}. Combining the asymptotic formulas for all the parts of Σ_7 which we *can* evaluate, we find that

$$\Sigma_7 \ge \frac{X^{1-\theta}}{2\log X}\left(c_7 - \ell_{11} - \ell_{12} + o(1)\right), \tag{13}$$

where ℓ_{11} and ℓ_{12} are the constants in the expected asymptotic formulas for Σ_{11} and Σ_{12}, respectively. For the record, we have

$$\begin{aligned}\ell_{11} &= \int_{1/2-\theta}^{\theta}\int_{1-2\theta-u}^{u} \omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2},\\ \ell_{12} &= \iiiint_{\mathcal{D}_{12}} \omega\Big(\frac{1-u_1-\cdots-u_4}{u_4}\Big)\frac{du_1\cdots du_4}{u_1u_2u_3u_4^2},\end{aligned}$$

where $\mathcal{D}_{12}$ is the set of points $(u_1,\dots,u_4)$ in $\mathbb{R}^4$ subject to the inequalities

$$1-3\theta < u_4 < \cdots < u_1 < \theta, \quad u_1+u_2<\theta, \quad \sum_{j\in J} u_j \notin [\theta, 1-2\theta] \text{ for any set } J \subseteq \{1,2,3,4\}.$$

Note that when $\frac{1}{4} < \theta < \frac{3}{10}$, the last condition is equivalent to the inequality $u_2+u_3+u_4 > 1-2\theta$.

Using both (12) and (13) to estimate the right side of (10), we replace (11) by

$$\sum_{p\sim X}\Psi(\alpha p) \ge \frac{X^{1-\theta}}{2\log X}\left(1-\ell_9-\ell_{11}-\ell_{12}+o(1)\right). \tag{14}$$

This bound is non-trivial when $\ell_9+\ell_{11}+\ell_{12}<1$. A quick numeric calculation shows that the last inequality holds when $\theta = 0.299$ but fails when $\theta = 0.2995$, so we barely miss a result for all $\theta < \frac{3}{10}$. Can we "push" a little further?...

When $\theta = \frac{3}{10}$, we have

$$\ell_9 \approx 0.473, \quad \ell_{11} \approx 0.606, \quad \ell_{12} < 0.001,$$

so the biggest "loss" comes from the trivial estimate for Σ_{11}. Can we estimate some part of Σ_{11} non-trivially? Indeed, we can. Let Σ_{13} and Σ_{14} be the parts of Σ_{11} subject to

$$p_1p_2^2 > X^{1-\theta} \quad \text{and} \quad p_1p_2^2 \le X^{1-\theta},$$

respectively. We shall eventually estimate Σ_{13} trivially, which will lead to a loss of

$$\ell_{13} = \int_{(1-\theta)/3}^{\theta}\int_{(1-\theta-u)/2}^{u} \omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2} \approx 0.113 \quad \text{when } \theta = 0.3.$$

We decompose Σ_{14} further by applying Buchstab's identity two more times, and we get

$$\begin{aligned}
\Sigma_{14} = &\sum_{p_1,p_2} \sum_{mp_1p_2\sim X} \psi(m,z)\Psi(\alpha m p_1 p_2) \\
&- \sum_{p_1,p_2} \sum_{z<p_3\le p_2} \sum_{mp_1p_2p_3\sim X} \psi(m,z)\Psi(\alpha m p_1 p_2 p_3) \\
&+ \sum_{p_1,p_2} \sum_{z<p_4\le p_3\le p_2} \sum_{mp_1\cdots p_4\sim X} \psi(m,p_4)\Psi(\alpha m p_1\cdots p_4),
\end{aligned} \tag{15}$$

where in all three sums the primes p_1 and p_2 satisfy the conditions

$$z < p_2 \le p_1 < X^{\theta}, \quad p_1p_2 > X^{1-2\theta}, \quad p_1p_2^2 \le X^{1-\theta}.$$

The first two sums on the right side of (15) lead to Type I/II sums, so they can be evaluated as before. Note that it is here that we need the condition $p_1p_2^2 \le X^{1-\theta}$ that we imposed on Σ_{14}. Without that assumption, there can be parts of the second sum on the right side of (15) which are not acceptable as Type I/II sums. We can also use Type II estimates to evaluate any part of the quintuple sum over $m, p_1, \dots, p_4$ where a subproduct of $p_1p_2p_3p_4$ lies in one of the ranges $[X^{\theta}, X^{1-2\theta}]$ or $[X^{2\theta}, X^{1-\theta}]$. Let us call the remaining parts of the quintuple sum (where no evaluation is possible) Σ_{15}, and let ℓ_{15} be the constant in the expected asymptotic formula for that sum. We then have

$$\Sigma_{11} \ge \frac{X^{1-\theta}}{2\log X}(\ell_{11} - \ell_{13} - \ell_{15} + o(1)), \tag{16}$$

where

$$\ell_{15} = \iiiint_{\mathcal{D}_{15}} \omega\Big(\frac{1-u_1-\cdots-u_4}{u_4}\Big)\frac{du_1\cdots du_4}{u_1u_2u_3u_4^2},$$

where $\mathcal{D}_{15}$ is the set of points $(u_1, \dots, u_4)$ in $\mathbb{R}^4$ subject to the inequalities

$$\begin{gathered}
1-3\theta < u_4 < \cdots < u_1 < \theta, \quad u_1 + 2u_2 < 1-\theta, \quad u_1 + u_2 > 1-2\theta, \\
\sum_{j\in J} u_j \notin [\theta, 1-2\theta] \cup [2\theta, 1-\theta] \text{ for any set } J \subseteq \{1,2,3,4\}.
\end{gathered}$$

When $\frac{2}{7} < \theta < \frac{3}{10}$, the condition on the subsums of $\sum_j u_j$ implies that

$$u_1+u_2+u_3+u_4 > 1-\theta, \quad u_2+u_3+u_4 > 1-2\theta, \quad u_1+u_2+u_3 < 2\theta. \tag{17}$$

In addition, $u_1, \dots, u_4$ must satisfy one of the following four sets of conditions:

(1) $u_1+u_3 < \theta$, $u_1+u_2 > 1-2\theta$;
(2) $u_1+u_4 < \theta$, $u_1+u_3 > 1-2\theta$;
(3) $u_2+u_3 < \theta$, $u_1+u_4 > 1-2\theta$;
(4) $u_3+u_4 < \theta$, $u_1+u_4 > 1-2\theta$, $u_2+u_3 > 1-2\theta$.

However, when $\frac{2}{7} < \theta < \frac{3}{10}$, three of these four sets of inequalities are incompatible with (17). For example, in case (4), we must have

$$u_1 = (u_1+u_2+u_3) - (u_2+u_3) < 4\theta - 1,$$

whence

$$u_1 + u_4 < 2u_1 < 8\theta - 2 < 1 - 2\theta,$$

a contradiction. In fact, only case (3) is consistent with the other conditions on $u_1, \dots, u_4$. Hence, $\mathcal{D}_{15}$ reduces to the set of points $(u_1, \dots, u_4)$ such that

$$\begin{gathered}
1-2.5\theta < u_1 < \theta, \quad (1-\theta-u_1)/3 < u_2 < u_1 + 3\theta - 1, \\
(1-\theta-u_1-u_2) < u_3 < \min(u_2, \theta-u_2), \quad 1-\theta-u_1-u_2-u_3 < u_4 < u_3.
\end{gathered}$$

After that, a quick numeric calculation shows that

$$\ell_{15} < 0.002 \quad \text{when } \theta = 0.3.$$

When $\theta = \frac{3}{10}$, we have reduced the loss from Σ_{11} from approximately 0.606 to less than 0.12. Therefore, we can use (16) to strengthen (14) to

$$\sum_{p\sim X} \Psi(\alpha p) \geq \frac{X^{1-\theta}}{2\log X}(1-\ell_9-\ell_{12}-\ell_{13}-\ell_{15}+o(1)), \tag{18}$$

and the constant on the right is ≥ 0.4 when $\theta = \frac{3}{10}$. In this way, we have proved that (1) has infinitely many solutions when $\theta < \frac{3}{10}$.

What happens when we cross over $\theta = \frac{3}{10}$? The immediate effect will be that a whole lot of additional cases will appear in the combinatorial analysis. For example, it is then possible to find points $(u_1,\dots,u_4)$ such that

$$1-3\theta < u_4 < u_3 < u_2, \quad u_2+u_3+u_4 < \theta.$$

This and other similar possibilities will make the quadruple integral ℓ_{15} much more complicated. However, since ℓ_{15} is a continuous function of θ, its value will not change drastically if we change the value of θ from 0.3 to $0.3+\varepsilon$. This allows us to claim the above result not only for $\theta < \frac{3}{10}$ but, in fact, for $\theta \leq \frac{3}{10}$. It is also clear that if we are willing to work our way through the more complicated combinatorial and numerical arguments that will follow, we can actually find some $\theta_0 > 0.3$ such that the numerical constant on the right side of (18) is positive for $\theta = \theta_0$ but it turns negative when $\theta = \theta_0 + 0.001$ (similarly to what happened earlier with $\theta = 0.298$). The work involved in that will, however, be too much to discuss here and will not be instructive anyways.

Lecture 5: Exceptional sets for sums of squares of primes. The circle method

In this lecture, we consider the following question.

Question. *Let x be a large real, and write*

$$E(X) = \#\{n \sim X : n \equiv 4 \pmod{24},\ n \neq p_1^2+\cdots+p_4^2\}.$$

For what $\theta > 0$ does the inequality $E(X) \ll X^\theta$ hold?

Let $\mathcal{E} = \mathcal{E}(X)$ denote the set of integers counted by $E(X)$. To estimate $E(X)$, we set

$$R(n) = \sum_{\substack{p_1^2+\cdots+p_4^2=n \\ p_i\sim N}} 1, \quad N = \tfrac{2}{3}X^{1/2}.$$

For each $n \sim X$, we have

$$R(n) = \int_0^1 f(\alpha)^4 e(-\alpha n)\,d\alpha, \quad f(\alpha) = \sum_{p\sim N} e(\alpha p^2).$$

Suppose that $1 \leq P \leq Q \leq X$, and define the sets of major and minor arcs by

$$\mathfrak{M} = \bigcup_{1\leq q\leq P} \bigcup_{\substack{1\leq a\leq q \\ (a,q)=1}} \Big(\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\Big), \quad \mathfrak{m} = \big[Q^{-1}, 1+Q^{-1}\big] - \mathfrak{M}.$$

First, we estimate the contribution from the major arcs, which produces the main term in the expected asymptotic formula for $R(n)$. For any fixed $A > 4$, we have

$$\int_{\mathfrak{M}} f(\alpha)^4 e(-\alpha n)\,d\alpha = \kappa_n N^2(\log N)^{-4} + O\big(N^2(\log N)^{-A}\big) \tag{19}$$

provided that PQ is "close" to N^2. Here, κ_n is a function of n which satisfies

$$1 \ll \kappa_n \ll \log\log X$$

for $n \equiv 4 \pmod{24}$ with $n \sim X$. When $P = (\log N)^{B_1}$ and $Q = N^2(\log N)^{-B_2}$, the proof of this result is an exercise using the Siegel–Walfisz theorem and partial summation. Using ideas from the last decade, we can take the major arcs considerably larger, though the proof is much longer and much more technical. Here, we simply report that we can choose any P and Q such that

$$P \leq N^{9/20-\varepsilon}, \quad Q \geq N^{31/20+\varepsilon}, \quad PQ \leq N^2.$$

The difficulty of the proof aside, we view (19) as the "easy" part of $R(n)$, similar to the contribution from the zeroth Fourier coefficient in the study of αp modulo one.

The estimation of the contribution from the minor arcs is harder, and we can obtain a bound only on average over n. We first note that, for any exceptional $n \in \mathcal{E}$, we have

$$-\int_{\mathfrak{m}} f(\alpha)^4 e(-\alpha n)\, d\alpha = \int_{\mathfrak{M}} f(\alpha)^4 e(-\alpha n)\, d\alpha \gg N^2 (\log N)^{-4}.$$

Hence,

$$-\int_{\mathfrak{m}} f(\alpha)^4 Z(\alpha)\, d\alpha = -\sum_{n\in\mathcal{E}} \int_{\mathfrak{m}} f(\alpha)^4 e(-\alpha n)\, d\alpha \gg E(X) N^2 (\log N)^{-4}, \tag{20}$$

where

$$Z(\alpha) = \sum_{n\in\mathcal{E}} e(-\alpha n).$$

We now estimate the left side of (20) from above. We have

$$\begin{aligned}\int_{\mathfrak{m}} f(\alpha)^4 Z(\alpha)\, d\alpha &\ll \big(\max_{\alpha\in\mathfrak{m}} |f(\alpha)|\big) \Big(\int_0^1 |f(\alpha)|^4\, d\alpha\Big)^{1/2} \Big(\int_0^1 |f(\alpha) Z(\alpha)|^2\, d\alpha\Big)^{1/2} \\ &\ll \big(\max_{\alpha\in\mathfrak{m}} |f(\alpha)|\big) N^{1+\varepsilon} \big(E(X)^{1/2} N^{1/2} + E(X)\big),\end{aligned} \tag{21}$$

for any fixed $\varepsilon > 0$. Therefore, if we assume an exponential sum bound of the form

$$\max_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll N^{1-\sigma}$$

for some fixed $\sigma > 0$, we can combine (20) and (21) to show that

$$E(X) \ll N^{-2} (\log N)^4 N^{5/2-\sigma+\varepsilon} E(X)^{1/2} \ll N^{1/2-\sigma+3\varepsilon/2} E(X)^{1/2},$$

whence, upon readjusting the choice of ε,

$$E(X) \ll X^{1/2-\sigma+\varepsilon}. \tag{22}$$

This reduces the problem of estimating $E(X)$ to that of estimating the exponential sum $f(\alpha)$ on the minor arcs. The best known estimate for $f(\alpha)$ has the form

$$f(a/q+\beta) \ll N^{1+\varepsilon}\big(q^{-1/2} + N^{-1/8}\big),$$

provided that β is "small" (essentially, $|\beta| < q^{-2}$). Therefore, if we choose P, Q in the definition of the minor arcs so that

$$N^{1/4} \ll P \ll N^{9/20-\varepsilon}, \quad Q = N^2 P^{-1},$$

the above argument will yield a bound (22) with $\sigma = \frac{1}{8}$. Can we use a sieve to do better than that?

LECTURE 6: EXCEPTIONAL SETS FOR SUMS OF SQUARES OF PRIMES. AN "EASY" ALTERNATIVE SIEVE

The standard estimation of $f(\alpha)$ on the minor arc uses ideas similar to the estimation of the exponential sum in Lecture 1. This time, we can show that: for Type II sums,

$$\sum_{m\sim M} \sum_{mn\sim N} a_m b_n e(\alpha m^2 n^2) \ll N^{1-\sigma+\varepsilon},$$

if $X^{2\sigma} \ll M \ll X^{1-4\sigma}$ or $X^{4\sigma} \ll M \ll X^{1-2\sigma}$; and for Type I sums,

$$\sum_{m\sim M} \sum_{mn\sim N} a_m e(\alpha m^2 n^2) \ll N^{1-\sigma+\varepsilon},$$

if $M \ll X^{1/2-\sigma}$. When $0 < \sigma < \frac{1}{8}$, the two ranges for M in our Type II sum estimate overlap, and we can combine the above estimates to obtain bounds for:

- Type I sums with $M \ll N^{1-2\sigma}$;
- Type II sums with $N^{2\sigma} \ll M \ll N^{1-2\sigma}$.

Thus, we can use Vaughan's identity in a similar way to Lecture 1 to derive a bound for $f(\alpha)$. When $\sigma > \frac{1}{8}$, however, the situation has changed compared to earlier, and not for the better. Indeed, when $\sigma > \frac{1}{8}$, we can estimate neither a Type I nor a Type II sum with $N^{1-4\sigma} \ll M \ll N^{4\sigma}$. Compare this to the situation at the beginning of Lecture 2, where the Type II information required by Vaughan's identity disappeared, but Type I sum bounds were still available.

In general, such breakdowns in analytic information pose significant problems to Harman's alternative sieve. Here, however, we can avoid those problems, since there is a "non-standard" Type I sum bound. We can combine that alternative Type I sum bound with the standard bound mentioned earlier to get

$$\sum_{m\sim M}\sum_{mn\sim N} a_m e(\alpha m^2 n^2) \ll N^{1-\sigma+\varepsilon},$$

for $M \ll N^{1-3\sigma}$. When $\sigma < \frac{1}{7}$, we can even combine this with our Type II sum bound to extend the range for M further to $M \ll N^{1-2\sigma}$. Once we have estimates for Type I sums with such large M, we can argue similarly to Lemma 1 to show that

$$\sum_{m\sim M}\sum_{mn\sim N} a_m \psi(n,z) e(\alpha m^2 n^2) \ll N^{1-\sigma+\varepsilon},$$

for $M \ll N^{1-2\sigma}$ and $z \le N^{1-6\sigma}$.

So, we now assume that $\frac{1}{8} < \sigma < \frac{1}{7}$ and try to apply the alternative sieve using:

- bounds for Type I/II sums with $M \ll N^{1-2\sigma}$ and $z \le X^{1-6\sigma}$;
- bounds for Type II sums with $N^{2\sigma} \ll M \ll N^{1-4\sigma}$ or $N^{4\sigma} \ll M \ll N^{1-2\sigma}$.

By Buchstab's identity,

$$\begin{aligned}
R(n) &= \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n\\ m,p_i\sim N}} \psi\big(m, N^{1/2}\big) \\
&= \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n\\ m,p_i\sim N}} \Big(\psi(m,z) - \sum_{z<p\le N^{1/2}} \psi(m/p,p)\Big) \\
&= \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n\\ m,p_i\sim N}} \Big(\psi(m,z) - \sum_{z<p\le N^{1/2}} \psi(m/p,z) + \sum_{\substack{z<q\le p\le N^{1/2}\\ pq^2\le N}} \psi(m/pq,q)\Big),
\end{aligned} \tag{23}$$

where $z = N^{1-6\sigma}$. We now want to evaluate the sums on the right side of (23) using the circle method, similarly to Lecture 5.

Let us consider, for example,

$$R_1(n) = \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n\\ m,p_i\sim N}} \psi(m,z).$$

We have

$$R_1(n) = \int_0^1 f(\alpha)^3 g_1(\alpha) e(-\alpha n)\, d\alpha, \qquad g_1(\alpha) = \sum_{m\sim N} \psi(m,z) e(\alpha m^2).$$

Similarly to (21), we have

$$\begin{aligned}
\int_{\mathfrak{m}} f(\alpha)^3 g_1(\alpha) Z(\alpha)\, d\alpha &\ll \big(\max_{\alpha\in\mathfrak{m}} |g_1(\alpha)|\big)\Big(\int_0^1 |f(\alpha)|^4\, d\alpha\Big)^{1/2}\Big(\int_0^1 |f(\alpha)Z(\alpha)|^2\, d\alpha\Big)^{1/2} \\
&\ll N^{1-\sigma+\varepsilon} N^{1+\varepsilon}\big(E(X)^{1/2}N^{1/2} + E(X)\big),
\end{aligned} \tag{24}$$

since $g_1(\alpha)$ is an admissible Type I/II sum.

Earlier, we declared the integral over the major arcs the "easy" one. That is only partially true, since its evaluation requires some delicate technical work (see Exercise 4). How is that work affected when we replace one of the exponential sums $f(\alpha)$ by $g_1(\alpha)$? The proof of (19) has two stages. The first, where the bulk of the effort is spent is to show that

$$\int_{\mathfrak{M}} \big(f(\alpha)^4 - f^*(\alpha)^4\big) e(-\alpha n)\, d\alpha \ll N^2(\log N)^{-A} \tag{25}$$

for any fixed $A > 4$. Here, $f^*(\alpha)$ is the expected major arc approximation to $f(\alpha)$, defined for $\alpha = a/q + \beta$ by

$$f^*(a/q+\beta) = \phi(q)^{-1}\Big(\sum_{h\in\mathbb{Z}_q^*} e(ah^2/q)\Big) \sum_{m\sim N} \frac{e(\beta m^2)}{\log m}.$$

Once we have (25), it is relatively easy to show that

$$\int_{\mathfrak{M}} f^*(\alpha)^4 e(-\alpha n)\, d\alpha \approx \kappa_n N^2 (\log N)^{-4}. \tag{26}$$

In the analysis of $R_1(n)$, (25) and (26) are replaced by

$$\int_{\mathfrak{M}} \big(f(\alpha)^3 g_1(\alpha) - f^*(\alpha)^3 g_1^*(\alpha)\big) e(-\alpha n)\, d\alpha \ll N^2(\log N)^{-A}, \tag{27}$$

$$\int_{\mathfrak{M}} f^*(\alpha)^3 g_1^*(\alpha) e(-\alpha n)\, d\alpha \approx c_1 \kappa_n N^2 (\log N)^{-4}, \tag{28}$$

respectively. Here, $g_1^*(\alpha)$ is the expected major arc approximation to $g_1(\alpha)$, defined for $\alpha = a/q + \beta$ by

$$g_1^*(a/q+\beta) = \phi(q)^{-1}\Big(\sum_{h\in\mathbb{Z}_q^*} e(ah^2/q)\Big) \sum_{m\sim N} \frac{e(\beta m^2)}{\log z}\omega\Big(\frac{\log m}{\log z}\Big),$$

where ω is Buchstab's function and $c_1 = \zeta^{-1}\omega(1/\zeta)$ with $\zeta = 1 - 6\sigma$. The constant c_1 in (28) is similar to the constants c_i, ℓ_j in Lectures 2–4 and is such that

$$\sum_{m\sim N} \frac{e(\beta m^2)}{\log z}\omega\Big(\frac{\log m}{\log z}\Big) \approx c_1 \sum_{m\sim N} \frac{e(\beta m^2)}{\log m} \tag{29}$$

for small β.

Between (27) and (28), (28) is the easier by far. Indeed, it is an easy exercise to deduce (28) from (26) and (29). The proof of (27) is another story. To explain the issues that arise, we need to sketch some of the highlights in the proof of (25). That proof, like the proof of the Bombieri–Vinogradov theorem, relies on two main facts about the primes:

(P_1) Primes are well-distributed in arithmetic progressions with small moduli. The analytic formulation of this fact, the Siegel–Walfisz theorem, states: For any fixed $A, B > 0$, and any primitive Dirichlet character χ with a modulus $\le (\log N)^B$, one has

$$\sum_{p\sim N} \chi(p) = \delta_\chi \sum_{m\sim N} (\log m)^{-1} + O\big(N(\log N)^{-A}\big),$$

where δ_χ is 1 when χ is the trivial character, and $\delta_\chi = 0$ otherwise.

(P_2) We can convert a sum over primes to a linear combination of Type I and Type II-like triple sums. More precisely, to achieve the major arcs mentioned above using the ideas outlined in Exercise 4, we require that the sum

$$\sum_{p\sim N} \Phi(p)$$

can be decomposed into a small number (say, $\ll (\log N)^c$) sums of the form

$$\sum_{u\sim U}\sum_{v\sim V}\sum_{uvw\sim N} \alpha_u \beta_v \gamma_w \Phi(uvw),$$

where $|\alpha_u| \le 1$, $|\beta_v| \le 1$, $|\gamma_w| \le 1$, $\max(U,V) \le N^{11/20}$, and either $UV \ge N^{27/35}$ or $\gamma_w = 1$ for all w.

The analogous properties of the sieve weight $\lambda(m) = \psi(m,z)$ are:

(S$_1$) For any fixed $A, B > 0$, and any non-principal Dirichlet character χ with a modulus $\le (\log N)^B$, and any interval $\mathcal{I} \subset (N/2, N]$, one has

$$\sum_{m \in \mathcal{I}} \chi(m)\lambda(m) \ll N(\log N)^{-A}.$$

(S$_2$) There is a smooth function $\delta(m)$ such that: for any fixed $A > 0$ and any interval $\mathcal{I} \subset (N/2, N]$, one has

$$\sum_{m \in \mathcal{I}} \lambda(m) = \sum_{m \in \mathcal{I}} \delta(m) + O\big(N(\log N)^{-A}\big).$$

(S$_3$) We can express $\lambda(m)$ as the linear combination of $\ll (\log N)^c$ arithmetic functions of the form

$$\sum_{u \sim U} \sum_{v \sim V} \sum_{uvw = m} \alpha_u \beta_v \gamma_w,$$

where $|\alpha_u| \le 1$, $|\beta_v| \le 1$, $|\gamma_w| \le 1$, $\max(U, V) \le N^{11/20}$, and either $UV \ge N^{27/35}$ or $\gamma_w = 1$ for all w.

Hypotheses (S$_1$) and (S$_2$) are used as a replacement for the Siegel–Walfisz theorem in the proof of (27); hypothesis (S$_3$) is used in the same way as (P$_2$) is used for primes. The truth of these three hypotheses for $\lambda(m) = \psi(m,z)$ and similar functions follows easily from the standard results on the distribution of primes.

There is one additional issue that arises in the presence of sieve weights. The transition from $f(\alpha)$ and $g_1(\alpha)$ to their major arc approximations $f^*(\alpha)$ and $g_1^*(\alpha)$ begins with an application of the orthogonality property of the Dirichlet characters. For example, if $\alpha = a/q + \beta$, with β small and $q \le P$, we write

$$\begin{aligned}\sum_{p \sim N} e(\alpha p^2) &= \sum_{h \in \mathbb{Z}_q^*} e(ah^2/q) \sum_{\substack{p \sim N \\ p \equiv h \pmod q}} e(\beta p^2) \\ &= \phi(q)^{-1} \sum_{\chi \bmod q} \sum_{h \in \mathbb{Z}_q^*} \bar{\chi}(h) e(ah^2/q) \sum_{p \sim N} \chi(p) e(\beta p^2).\end{aligned}$$

This is justified, because when $P < N/2$, the primes $p \sim N$ fall only in residue classes $h \bmod q$ with $(h,q) = 1$. The latter, however, is not true for the support of the sieve weights. For example, if p_0 is a prime with $z < p_0 \le P$, then there will be moduli q, $z < q \le P$ which are divisible by p_0, and for such moduli $g_1(\alpha)$ will include terms with $(m,q) = p_0$. When $z \ge N^\sigma$, it is relatively easy to solve this problem. Define the function $\theta(m,\alpha)$ on the major arcs by

$$\theta(m,\alpha) = \begin{cases} 0 & \text{if } \alpha = a/q + \beta \text{ and } (m,q) \ge N^\sigma, \\ 1 & \text{otherwise;} \end{cases}$$

then set

$$\tilde{g}_1(\alpha) = \sum_{m \sim N} \psi(m,z)\theta(m,\alpha)e(\alpha m^2), \qquad h_1(\alpha) = g_1(\alpha) - \tilde{g}_1(\alpha).$$

When $z \ge N^\sigma$ and $\alpha = a/q + \beta$, the sum $\tilde{g}_1(\alpha)$ is supported on integers with $(m,q) = 1$, so

$$\begin{aligned}\tilde{g}_1(\alpha) &= \sum_{h \in \mathbb{Z}_q^*} e(ah^2/q) \sum_{\substack{m \sim N \\ m \equiv h \pmod q}} \psi(m,z)\theta(m,\alpha)e(\beta m^2) \\ &= \phi(q)^{-1} \sum_{\chi \bmod q} \sum_{h \in \mathbb{Z}_q^*} \bar{\chi}(h) e(ah^2/q) \sum_{m \sim N} \chi(m)\psi(m,z)e(\beta m^2).\end{aligned} \tag{30}$$

Note that after the characters have been introduced the weights $\theta(m,\alpha)$ become trivial and can be dropped. Thus, we can use the generating function $\tilde{g}_1(\alpha)$ (note that this function is not an exponential sum) in place of $g_1(\alpha)$ in the treatment of the major arcs to show that

$$\int_{\mathfrak{M}} \big(f(\alpha)^3 \tilde{g}_1(\alpha) - f^*(\alpha)^3 g_1^*(\alpha)\big) e(-\alpha n)\, d\alpha \ll N^2 (\log N)^{-A}$$

instead of (27). This leaves

$$\int_{\mathfrak{M}} f(\alpha)^3 h_1(\alpha) e(-\alpha n)\, d\alpha.$$

This last integral can be estimated, on average over n, in a similar way to the integral over the minor arcs. Indeed, for $\alpha = a/q + \beta$,

$$|h_1(\alpha)| \le \sum_{\substack{p \mid q \\ p \ge N^\sigma}} \sum_{\substack{m \sim N \\ p \mid m}} 1 \ll N^{1-\sigma},$$

whence

$$\int_{\mathfrak{M}} f(\alpha)^3 h_1(\alpha) Z(\alpha)\, d\alpha \ll N^{1-\sigma+\varepsilon} N^{1+\varepsilon}\big(E(X)^{1/2}N^{1/2} + E(X)\big).$$

These tricks suffice to deal with the major arcs when $z \geq N^{\sigma}$. When $z < N^{\sigma}$, we must include additional terms to the right side of (30) and things get considerably more complicated (see [Harman & Kumchev, J. Number Theory 130 (2010)]). Observe that the condition $z \geq N^{\sigma}$ means that $\sigma < \frac{1}{7}$, so it is currently satisfied.

We conclude that when $\frac{1}{8} < \sigma < \frac{1}{7}$, we have

$$R_1(n) \approx c_1 \kappa_n N^2 (\log N)^{-4},$$

with $\ll X^{1/2-\sigma+\varepsilon}$ exceptions for $n \sim X$. We then obtain similar results for

$$R_2(n) = \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n \\ m,p_i\sim N}} \sum_{z<p\leq N^{1/2}} \psi(m/p, z)$$

and

$$R_3(n) = \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n \\ m,p_i\sim N}} \sideset{}{^\dagger}\sum_{\substack{z<q\leq p\leq N^{1/2} \\ pq^2\leq N}} \psi(m/pq, q)\Big),$$

where in $R_3(n)$ the primes p and q are such that either p, or q, or pq lies in the ranges $[N^{2\sigma}, N^{1-4\sigma}]$ or $[N^{4\sigma}, N^{1-2\sigma}]$. Consequently, by (23),

$$R(n) \geq (1-\ell)\kappa_n N^2 (\log N)^{-4},$$

with $\ll X^{1/2-\sigma+\varepsilon}$ exceptions for $n \sim X$, where

$$\ell = \Big(\int_{1-4\sigma}^{1/2}\int_{1-2\sigma-u}^{(1-u)/2} + \int_{1/2-2\sigma}^{2\sigma}\int_{1-4\sigma-u}^{u}\Big)\omega\Big(\frac{1-u-v}{v}\Big)\frac{dv\,du}{uv^2}.$$

When $\sigma = \frac{1}{7}$, we have $\ell < 0.3$, so this suffices to show that

$$E(X) \ll X^{5/14+\varepsilon}.$$

We conclude this lecture with the remark that the limitation $\sigma < \frac{1}{7}$ appeared in two places above: in the proof of the extended Type I/II bound and in the treatment of the major arcs (see (30)). Neither of these restrictions is critical and can be overcome when $\sigma > \frac{1}{7}$ at the expense of a much messier proof: see [Harman & Kumchev, J. Number Theory 130]. The fix, however, is only temporary and another obstruction appears when $\sigma = \frac{3}{20}$, though that obstruction seems to be non-critical too.

LECTURE 7: EXCEPTIONAL SETS FOR SUMS OF SQUARES OF PRIMES. A "HARD" ALTERNATIVE SIEVE

In the last lecture, we were lucky to have the alternative Type I bound. In many applications to additive problems, we have no such luck. To demonstrate how one can deal with such situations, we go back to the same problem, but this time we shall make the problem harder by ignoring the alternative Type I sum bound. Thus, we now assume that $\frac{1}{8} < \sigma < \frac{1}{7}$ and try to apply the alternative sieve using:

- bounds for Type I/II sums with $M \ll N^{1-4\sigma}$ and $z \leq X^{1-6\sigma}$;
- bounds for Type II sums with $N^{2\sigma} \ll M \ll N^{1-4\sigma}$ or $N^{4\sigma} \ll M \ll N^{1-2\sigma}$.

Under these constraints, we can no longer evaluate all of $R_2(n)$. Yet, since $R_2(n)$ contributes a negative amount to $R(n)$, we cannot ignore its "bad" part (where $p > N^{1-4\sigma}$) either. So far, we have

$$R(n) \geq R_1(n) - R_2'(n) - R_2''(n) + R_3(n), \tag{31}$$

where $R_2'(n)$ and $R_2''(n)$ are the parts of $R_2(n)$ subject to $p \leq N^{1-4\sigma}$ and $p > N^{1-4\sigma}$, respectively. The methods from the previous section can be used to evaluate $R_1(n), R_2'(n)$ and $R_3(n)$. To complete the estimation of $R(n)$, we combine those evaluations with an upper bound for $R_2''(n)$.

To estimate $R_2''(n)$ from above, we shall apply the same sieve ideas to one of the other primes. Let

$$\lambda_2(m) = \sum_{N^{1-4\sigma}<p\le N^{1/2}} \psi(m/p,p).$$

Then

$$\begin{aligned}
R_2''(n) &= \sum_{\substack{m^2+p_1^2+p_2^2+p_3^2=n\\ m,p_i\sim N}} \lambda_2(m) = \sum_{\substack{m^2+k^2+p_1^2+p_2^2=n\\ m,k,p_i\sim N}} \lambda_2(m)\psi(k,N^{1/2})\\
&= \sum_{\substack{m^2+k^2+p_1^2+p_2^2=n\\ m,k,p_i\sim N}} \lambda_2(m)\Big(\psi(k,z) - \sum_{z<p\le N^{1/2}} \psi(k/p,p)\Big)\\
&\le \sum_{\substack{m^2+k^2+p_1^2+p_2^2=n\\ m,k,p_i\sim N}} \lambda_2(m)\Big(\psi(k,z) - \sum_{z<p\le N^{1/2-2\sigma}} \psi(k/p,p) - \sum_{N^{2\sigma}\le p\le N^{1-4\sigma}} \psi(k/p,p)\Big)\\
&= R_{2,1}''(n) - R_{2,2}''(n) - R_{2,3}''(n), \quad \text{say.}
\end{aligned}$$

We can evaluate $R_{2,1}''(n)$ and $R_{2,3}''(n)$ similarly to $R_1(n)$ using a Type I/II and a Type II sum bound, respectively.

We now give a further decomposition of $R_{2,2}''(n)$. We have

$$\begin{aligned}
R_{2,2}''(n) &= \sum_{\substack{m^2+k^2+p_1^2+p_2^2=n\\ m,k,p_i\sim N}} \lambda_2(m) \sum_{z<p\le N^{1/2-2\sigma}} \Big(\psi(k/p,z) - \sum_{z<q\le p\le N^{1/2-2\sigma}} \psi(k/pq,q)\Big)\\
&= R_{2,4}''(n) - R_{2,5}''(n);
\end{aligned}$$

here q denotes a prime. Inserting this into the decomposition of $R_2''(n)$, we find that

$$R_2''(n) \le R_{2,1}''(n) - R_{2,3}''(n) - R_{2,4}''(n) + R_{2,5}''(n).$$

Observe that when $\sigma < \frac{1}{7}$, we can evaluate $R_{2,5}''(n)$ using our Type II sum bound, since

$$N^{2\sigma} \le z^2 < pq \le N^{1-4\sigma}.$$

When we combine the upper bound for $R_2''(n)$ with the evaluations for $R_1(n)$, $R_2'(n)$ and $R_3(n)$, we deduce using (31) that

$$R(n) \ge (1-\ell-\ell_2(1+\ell_2+\ell_3))\kappa_n N^2(\log N)^{-4},$$

with $\ll X^{1/2-\sigma+\varepsilon}$ exceptions for $n\sim X$. Here,

$$\begin{aligned}
\ell_2 &= \int_{1-4\sigma}^{1/2} \omega\Big(\frac{1-u}{u}\Big)\frac{du}{u^2} = \log\Big(\frac{4\sigma}{1-4\sigma}\Big);\\
\ell_3 &= \int_{1/2-2\sigma}^{2\sigma} \omega\Big(\frac{1-u}{u}\Big)\frac{du}{u^2} \le 0.66 \quad \text{when } \sigma = \tfrac{1}{7}.
\end{aligned}$$

Hence, when $\sigma < \frac{1}{7}$, we have

$$R(n) \ge 0.14\kappa_n N^2(\log N)^{-4},$$

with $\ll X^{1/2-\sigma+\varepsilon}$ exceptions for $n\sim X$. We have recovered our earlier result with a weaker lower bound for the number of representations.

Exercise 1. **(a)** Let $0 < \Delta < \frac{1}{2}$. Let Φ be a smooth function supported in $[-\Delta, \Delta]$, such that

$$0 \le \Phi(x) \le 1, \qquad \int_{\mathbb{R}} \Phi(x)\, dx = \Delta,$$

and define $\Psi(x) = \Phi(\|x\|)$. Prove that Ψ has a Fourier expansion of the form

$$\Psi(x) = \Delta + \sum_{h \ne 0} c_h e(hx),$$

with Fourier coefficients satisfying

$$c_h = \hat{\Phi}(h) \ll_k \frac{\Delta}{\big(1 + \Delta|h|\big)^k} \qquad (k \ge 1).$$

(b) Let X be a large real and Ψ be as in part (a). Suppose that (a_n) is a real sequence and (ξ_n) is a sequence of complex numbers such that $|\xi_n| \le n^A$ for some $A > 0$. Then, for any fixed $\varepsilon > 0$,

$$\sum_{n \le X} \xi_n \Psi(a_n) = \Delta \sum_{n \le X} \xi_n + \sum_{0 < |h| \le H} c_h \sum_{n \le X} \xi_n e(ha_n) + O_{\varepsilon, A}(\Delta X^{-1}),$$

where $H = \Delta^{-1-\varepsilon} X^{\varepsilon}$.

Exercise 2. For $z \ge 2$, let $\psi(n,z)$ be defined by

$$\psi(n,z) = \begin{cases} 1 & \text{if } n \text{ has no prime divisor } p \text{ with } p \le z, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that x is a large real and $0 < \delta \le \zeta \le 1 - \delta$ for some fixed $\delta > 0$, and set $k = \lfloor \delta^{-1} \rfloor$. Show that

$$\sum_{n \le x} \psi\big(n, x^{\zeta}\big) = \sum_{n \le x} \psi\big(n, x^{1/k}\big) + \sum_{x^{\zeta} < p \le x^{1/k}} \sum_{mp \le x} \psi(m,p).$$

Use this identity, the Prime Number Theorem, and induction on k to prove that, for any fixed $A > 0$,

$$\begin{aligned} \sum_{n \le x} \psi\big(n, x^{\zeta}\big) &= \frac{1}{\log z} \sum_{n \le x} \omega\Big(\frac{\log n}{\log z}\Big) + O\big(x(\log x)^{-A}\big) \\ &= \frac{x}{\log x}\big(\zeta^{-1}\omega(1/\zeta) + O\big((\log x)^{-1}\big)\big), \end{aligned}$$

where ω is Buchstab's function.

Exercise 3. **(a)** Let $\Phi : \mathbb{N} \to \mathbb{C}$ satisfy $|\Phi(x)| \le X$, let $M, N \ge 2$, and define the bilinear form

$$\mathcal{A}(M,N) = \sum_{\substack{m \sim M \\ m < n}} \sum_{n \sim N} a_m b_n \Phi(mn),$$

where $|a_m| \le 1$, $|b_n| \le 1$. Use Perron's formula to show that

$$\mathcal{A}(M,N) = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} \sum_{m \sim M} \sum_{n \sim N} a_m b_n \Phi(mn) \Big(\frac{m}{n - 1/2}\Big)^s \frac{ds}{s} + O(1),$$

for $c > 0$ and a suitable choice of T. Use this representation with a suitable choice of $c > 0$ to show that there exist coefficient sequences (a'_m) and (b'_n) such that $|a'_m| \ll |a_m|$, $|b'_n| \ll |b_n|$, and

$$\mathcal{A}(M,N) \ll L \Bigg| \sum_{m \sim M} \sum_{n \sim N} a'_m b'_n \Phi(mn) \Bigg| + 1,$$

where $L = \log(2MNX)$.

(b) Let $\Phi : \mathbb{N} \to \mathbb{C}$ satisfy $|\Phi(x)| \le X$, let $M, N \ge 2$, with $\log N \ll \log M$, and define the bilinear form

$$\mathcal{B}(M,N) = \sum_{m \sim M} \sum_{n \sim N} a_m \psi(n,m) \Phi(mn),$$

where $|a_m| \le 1$. Use Buchstab's identity to show that

$$\mathcal{B}(M,N) = \sum_{m \sim M} \sum_{n \sim N} a_m \psi(n, M/2) \Phi(mn) - \sum_{m \sim M} \sum_{M/2 < p \le m} \sum_{pq \sim N} a_m \psi(q,p) \Phi(mpq).$$

Then apply part (a) to deduce that there exist coefficient sequences (a'_m) and (b'_n) such that $|a'_m| \ll |a_m|$, $|b'_n| \ll 1$, and

$$\mathcal{B}(M,N) \ll L \left| \sum_{m\sim M} \sum_{n\sim N} a'_m b'_n \Phi(mn) \right| + 1,$$

where $L = \log(2MNX)$.

Exercise 4 (Hard)**.** Use Theorem 1.1 in [Choi & Kumchev, Acta Arith. 123 (2006), 125–142] instead of Lemma 5.6 in [Liu & Zhan, New Developments in the Additive Theory of Prime Numbers] to prove Theorem 6.1 in [Liu & Zhan] for $s = 4$ and any fixed $\theta < \frac{9}{20}$.

Exercise 5 (Hard)**.** Use Theorem 1.1 in [Choi & Kumchev] instead of Lemma 5.6 in [Liu & Zhan] to prove Theorem 6.1 in [Liu & Zhan] for $s = 3$ and any fixed $\theta < \frac{9}{20}$.
[Hint: When $s = 3$, show that the right side of (6.16) in [Liu&Zhan] can be replaced by $(n, r_0)^{1/2} r_0^{-1+\varepsilon} (\log x)^c$. Then follow the argument from Exercise 4 with some extra care. Lemma 1 in [Harman & Kumchev, J. Number Theory 130 (2010), 1969–2002] is useful to take full advantage of this sharper version of (6.16) in [Liu & Zhan].]