Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§8 Application of the Selberg Trace Formula (March 5, 2009)

Weyl law Classical case: Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain. The Laplace operator is given by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and the eigenfunctions are the solutions of $(\Delta + \lambda)f = 0$ with boundary conditions. The corresponding eigenvalues (with multiplicities) are $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Question. What is the asymptotic behavior of the $\lambda_j{}'s$? Weyl showed that

$$\sharp\{\lambda_j \leq T\} \sim \frac{\text{area}(\Omega)}{4\pi} T.$$

Much more generally, if M is a compact surface with Riemann metric, Δ is the Laplace operator which is unbounded on $L^2(\Gamma\backslash\mathbb{H})$ with discrete spectrum. Then we have

$$\sharp\{\lambda_j \leq T\} \sim \frac{\text{area}(\Omega)}{4\pi} T.$$

If M is not compact, then we can not expect such an asymptotic.

What happens for $\Gamma\backslash\mathbb{H}$ (say $\Gamma = SL_2(\mathbb{Z})$)? Problem: discrete eigenvalues are embedded in the continuous ones. Discrete spectrum : $\lambda_j = \frac{1}{4} + t_j^2,\ \ t_j \in \mathbb{R} \cup [-\frac{i}{2}, \frac{i}{2}]$, continuous spectrum : $\frac{1}{4} + r^2,\ r \in \mathbb{R}$. How to separate the discrete spectrum from the continuous spectrum?

Selberg Trace Formula gives an expression

$$\sum_{j=0}^{\infty} h(t_j) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'}{\phi}(\frac{1}{2} + ir) h(r) \mathrm{dr}$$

for any "nice" test function h ($\lambda_j = \frac{1}{4} + t_j^2$). Roughly, we would like

$$h(t) = \chi_{[-T,T]\cup[-\frac{i}{2},\frac{i}{2}]}.$$

That would give

$$\sharp\{\lambda_j \leq T\} - \frac{1}{4\pi} \int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2} + ir) h(r) \mathrm{dr}$$

with $\lambda_j \leq T^2 + \frac{1}{4}$. Such h is not permissible in the trace formula, we need a smooth substitute.

We will follow the method of Hörmander as used by Duistermint-Kolk-Vardarajar.

Recall STF, for $SL^2(\mathbb{Z})$, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty} h(t_j) - \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)\mathrm{d}r \\
&= \frac{\mathrm{area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h(r)r\tanh\pi r\mathrm{d}r \\
&+ 2\sum_{P}\sum_{l=1}^{\infty}\frac{g(l\log p)}{p^{\frac{l}{2}}-p^{-\frac{l}{2}}}\log p \\
&+ \sum_{\mathcal{R}}\sum_{0<l\leq m}\frac{1}{2m\sin\frac{\pi l}{m}}\int_{\mathbb{R}} h(r)\frac{\cosh\pi(1-\frac{2l}{m})r}{\cosh\pi r}\mathrm{d}r \\
&+ \frac{h(0)}{4}(1-\phi(\frac{1}{2})) - g(0)\log 2 - \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\mathrm{d}r,
\end{aligned}$$

where $h(t) = \int_{\mathbb{R}} e^{irt}g(r)\mathrm{d}r$ is an entire function.

Fix an even function $g \in C_c^\infty(\mathbb{R})$ such that

(i) $g(0) = 1$;

(ii) $h \geq 0$ on $\mathbb{R} \cup [-\frac{i}{2}, \frac{i}{2}]$;

(iii) $h > 0$ on $[-1, 1]$.

For any $t \in \mathbb{R}$, we consider $g_t(x) = \cos(tx)g(x)$, $h_t(r) = \frac{1}{2}[h(t-r)+h(t+r)]$, we want to use the trace formula for (g_t, h_t), for any t.

Lemma.

$$\frac{\phi'}{\phi}(\frac{1}{2}+it) \geq c$$

for some c and all $t \in \mathbb{R}$.

Proof. Applying Maass-Selberg relation

$$0 \leq ||E^T(\frac{1}{2}+it)||^2 = -\frac{\phi'}{\phi}(\frac{1}{2}+it) + 2\log T + \mathrm{Im}\frac{\phi(\frac{1}{2}-it)T^{2it}}{t}$$

and $|\phi(\frac{1}{2}+it)| = 1$, we get the result.

Spectral side:

$$[\sum_{t_j:\ |t_j-t|\leq 1} 1 - \int_{t-1}^{t+1}\frac{\phi'}{\phi}(\frac{1}{2}+ir)\mathrm{d}r]\min_{[-1,1]} h + O(1).$$

This just follows from the non-negativity of h and Lemma 1.

Geometric side:

Main term is

$$\frac{\text{area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h_t(r) r\tanh\pi r \mathrm{d}r \leq \frac{\text{area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h(t-r) r\mathrm{d}r$$
$$= \frac{\text{area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h(r)(t-r)\mathrm{d}r$$
$$\ll t.$$

Hyperbolic contribution: Only fixed $\sharp$ of terms. It is independent of t, so the contribution is $O(1)$. (By shrinking the support of g, no contribution at all.)

Elliptic contribution: Using the trivial bound

$$\Big|\frac{\cosh\pi(1-\frac{2l}{m})}{\cosh\pi r}\Big| \leq 1,$$

we get the contribution is $O(1)$. In fact, the elliptic contribution is $O(e^{-\alpha t})$.

The remaining contribution: $h_t(0) = h(t) = O(1),\ g_t(0) = g(t) = O(1)$ and

$$\int_{\mathbb{R}} h_t(r)\frac{\Gamma'}{\Gamma}(1+ir)\mathrm{d}r = \int_{\mathbb{R}} h(t-r)\frac{\Gamma'}{\Gamma}(1+ir)\mathrm{d}r$$
$$= \int_{\mathbb{R}} h(r)\frac{\Gamma'}{\Gamma}(1+ir+it)\mathrm{d}r$$
$$\ll \int_{\mathbb{R}} \frac{\log|r+t+2|}{(|r|+1)^2}\mathrm{d}r$$
$$\ll \log t$$

where h is rapidly decreasing, i.e., for any $N > 0,\ h(t) \ll (1+|t|)^{-N}$, and we also use Stirling's formula

$$\frac{\Gamma'}{\Gamma}(1+it) \ll \log|r|.$$

Conclusion:

$$\sum_{|t_j-t|\leq 1} - \int_{t-1}^{t+1}\frac{\phi'}{\phi}(\frac{1}{2}+ir)\mathrm{d}r \ll t + O(\log t).$$

This gives a local estimate for the spectrum.

Lemma1 Let μ be a measure on $\mathbb{R}$, s.t. $\mu([-t,t+1]) \ll t$. Then

$$\int_{-T}^{T}\int_{\mathbb{R}} h(t-r)d\mu(r)dt = \mu([-T,T]) + O(T).$$

Proof. Change the order of the integration in the LHS of the formula in Lemma 1, and we get

$$\text{LHS} = \int_{\mathbb{R}}\Big[\int_{-T}^{T} h(t-r)dt\Big]d\mu(r) = \mu([-T,T]) + O(T).$$

Since h is rapidly decreasing, we have

$$\int_{-T}^{T} h(t-r)dt = \begin{cases} 1+O\Big((1+|T-r|)^{-N}\Big), & |r| \le T, \\ O\Big((1+|T-r|)^{-N}\Big), & |r| > T. \end{cases}$$

So

$$\text{LHS} = \int_{-T}^{T} d\mu(r) + \int_{\mathbb{R}} O\Big((1+|T-r|)^{-N}\Big) d\mu(r).$$

The first integral is $\mu([-T,T])$, and the second integral $\ll \sum \frac{\mu([T+n,T+n+1])}{n^N} \ll T$.

In the trace formula we have expressions of the form

$$\int_{-T}^{T} \int_{\mathrm{R}} h(t-r) d\mu(r) dt$$

where $d\mu(r)$ has following forms : $d\mu(r) = r\tanh \pi r dr$, $u = \sum_j \delta_{t_j}$, $d\mu(r) = -\frac{\phi'}{\phi}(\frac{1}{2}+ir)$, $d\mu(r) = \frac{\Gamma'}{\Gamma}(1+ir)$. for the last two cases, we can deal by using local estimate following from previous discussions.

Using the Lemma, we get that the spectral side is

$$\sharp\{t_j \le T\} - \int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2}+ir)dr + O(T),$$

the main term of geometric side is

$$\begin{aligned} \int_{-T}^{T} r\tanh \pi r dr + O(T) &= \int_{-T}^{T} r(1+O(e^{-\pi r}))dr + O(T) \\ &= T^2 + O(T), \end{aligned}$$

parabolic term is

$$O(T) + \int_{-T}^{T} \frac{\Gamma'}{\Gamma}(1+ir)dr = T\log T + O(T),$$

hyperbolic and elliptic terms are $O(T)$ by local estimate following from previous discussions. Hence we have

$$\overset{\text{discrete}}{\sharp\{t_j \le T\}} - \overset{\text{continuous}}{\int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2}+ir)dr} = \frac{\text{area}(\Gamma\backslash \mathrm{H})}{4\pi}T^2 + T\log T + O(T).$$

Questions

1. What is the meaning of the continuous term?
2. Is it negligible?
3. Can we improve the error term?

Weierstrass factorization. For $n \in \mathbb{N}$, define

$$E_n(z) = \exp(\sum_{j=1}^{n} \frac{z^j}{j})$$

(exp of truncation of power series of $-\log(1-z)$). For any entire function, there is a function n from the set of zeros of f to $\mathbb{N}$ and a entire function g, s.t.

$$f(z) = e^{g(z)} z^{m(0)} \prod_{\eta \neq 0} \Big[(1 - \frac{z}{\eta})^{m(\eta)} E_{n(\eta)}(\frac{z}{\eta}) \Big]$$

where $m(\eta)$ is the multiplicity of η.

Define: An entire function is of finite order if $\exists\ n$ s.t.

$$f(z) \ll e^{|z|^n}.$$

Fact: f is of finite order $\Leftrightarrow \exists\ n$ s.t.

$$f(z) = e^{g(z)} \prod_{\eta} \Big[(1 - \frac{z}{\eta})^{m(\eta)} E_n(\frac{z}{\eta}) \Big], \tag{0.1}$$

where $g(z)$ is a polynomial.

The minimal n for which the product converges is called the order of f. It is closely related to

$$\overline{\lim} \frac{\log \Big(\sum_{|\eta| < \mathbb{R}} m(\eta) \Big)}{\log \mathbb{R}}.$$

For this n we get Hadamard canonical product.

We say that a meromorphic function f is of finite order, if $\exists\ g$, h entire function of finite order, s.t.

$$f(z) = \frac{g(z)}{h(z)}.$$

Equivalently, f is of finite order iff $\exists\ n \in \mathbb{N}$ and a polynomial $g(z)$ s.t. (0.1) holds ($m(\eta)$ could be negative).

EX. We have

$$\Gamma(z) = z^{-1} e^{\gamma z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)^{-1} e^{\frac{z}{n}},$$

where $\gamma =$ Euler's constant.

Blaschke product: Let $\{\eta_n\}_{n=1}^{\infty}$ be complex numbers, $\mathrm{Re}(\eta_n) > 0$ and

$$\sum \frac{\mathrm{Re}(\eta_n)}{|\eta_n|^2} < \infty.$$

Define

$$f(z)=\prod_{\eta}\frac{z-\eta}{z+\bar{\eta}}.$$

EX. $f(z)$ converges and meromorphic, whose zeros are $\{\eta_n\}$ and poles are $\{-\bar{\eta}_n\}$. We also have $|f(z)|=1$ for $\mathrm{Re}(z)=0$. Note that f does not need to be of finite order.

Conversely, suppose that f is meromorphic, holomorphic near $\mathrm{Re}(s)\geq 0$, $|f(s)|=1$ for $\mathrm{Re}(s)=0$ and f is of finite order. Then

$$f(s)=e^{g(s)}\prod_{\eta}\frac{s-\eta}{s+\bar{\eta}},$$

where η goes through all zeros of f, $g(s)$ is a polynomial satisfying $\mathrm{Re}g(s)=0$ for $\mathrm{Re}(s)=0$, and we have

$$\sum_{\eta \text{ zeros}}\frac{\mathrm{Re}(\eta_n)}{|\eta_n|^2}<\infty.$$

If moreover,

$$f(s)\sim\frac{1+o(1)}{\sqrt{s}} \text{ for } \mathrm{Re}(s)>2,$$

in particular, f has no zeros for $\mathrm{Re}(s)>2$, then g is constant.

$\phi(s)$ does not quite have these properties. But

$$\phi(s)\prod_{j=1}^{\infty}\frac{s-s_j}{1+s-s_j}$$

has these properties (even for Γ non-arithmetic), where $\{s_j\}$ are poles for $\mathrm{Re}(s)>\frac{1}{2}$. Therefore

$$\frac{\phi'}{\phi}=\sum_{s_j}\frac{1}{s-s_j}-\frac{1}{s-1+\bar{s_j}},$$

and $\sum\frac{\mathrm{Re}(1-s_j)}{|s_j|^2}<\infty$.

Note that $\mathrm{Re}(s)=\frac{1}{2}$,

$$\frac{1}{s-s_j}-\frac{1}{s-1+\bar{s_j}}\geq 0 \text{ if } \mathrm{Re}(s)<\frac{1}{2},$$

$$\frac{\phi'}{\phi}\geq O(|s|^2) \text{ on } \mathrm{Re}(s)=\frac{1}{2}$$

and

$$\int_{-T}^{T}\frac{\phi'}{\phi}(\frac{1}{2}+ir)dr=\sharp \text{ of poles } \phi \text{ with } \mathrm{Im}\leq T+O(T),$$

noting that $\sharp$ of poles ϕ with Im $\leq T = \sharp$ of zeros ϕ with Im $\leq T$ (We use that ϕ is of finite order, this comes from general theory).

In the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$,

$$\phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)}$$

is of order 1, and

$$\int_{-T}^{T} \frac{\phi'}{\phi} \sim T \log T.$$

Thus,

$$\sharp\{t_j \leq T\} = \frac{\mathrm{area}(\Gamma\backslash \mathrm{H})}{4\pi} T^2 + O(T \log T).$$

However, for a generic Γ one expects that $\sharp t_j$ is finite. In particular, ϕ is of order 2.

References