Graduate Algebra

Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

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1, Let $\rho: G \to GL_n(\mathbb{C})$ be a representation of a finite group G.
(i), Prove that $\pi: g \mapsto \det \rho_g$ is one dimensional representation.
(ii), Prove that the quotient group $G/\ker \pi$ is abelian.

2, Let S_n be the symmetric group on n letters. Prove that
(i), $S_3 = \langle (12), (123) \rangle$.
(ii), Denote by

$$\rho_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\rho: S_3 \to GL_2(\mathbb{C})$ is a representation.
(iii), The repr ρ is irreducible.

3, Let $\rho: G \to GL(V)$ be a representation of a finite group G. Define the fixed subspace

$$V^G = \{v \in V \mid \rho_g v = v, \forall g \in G\}.$$

(i), Show that V^G is a G-invariant subspace of V.
(ii), Show that, for all $v \in V$, we have

$$\frac{1}{|G|} \sum_{h \in G} \rho_h v \in V^G.$$

(iii), Show that if $v \in V^G$, then

$$\frac{1}{|G|} \sum_{h \in G} \rho_h v = v.$$

4, Let G be an abelian group. Then any irreducible representation of G has degree one.

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5, Let $\rho: G \to GL(V)$ be a representation of G with character χ and let V^* be the dual space of V, i.e., $V^* = \mathrm{Hom}(V, \mathbb{C})$, the linear forms on V. For $v \in V, v^* \in V^*$, let $\langle v, v^* \rangle = v^*(v)$. Show that there exists a unique linear representation $\rho^*: G \to GL(V^*)$, such that

$$\langle \rho_g v, \rho_g^* v^* \rangle = \langle v, v^* \rangle, \quad \forall g \in G, v \in V, v^* \in V^*.$$

This is called the contragredient representation of ρ and its character is $\bar{\chi}$.

6, Let $\rho: G \to GL(V)$ be an irreducible representation. Let

$$Z(G) = \{g \in G \,|\, gh = hg, \, \forall h \in G\}$$

be the center of G. Show that if $g \in Z(G)$, then $\rho_g = \lambda I$ for some $\lambda \in \mathbb{C}^{\times}$.

7, Let χ be a nontrivial irreducible character of G. Show that

$$\sum_{g \in G} \chi(g) = 0.$$

8, Let $\rho : S_n \to GL_n(\mathbb{C})$ be a representation given by defining $\rho_\sigma(e_i) = e_{\sigma(i)}$ on the standard basis $\{e_1, \cdots, e_n\}$ for $\mathbb{C}^n$.

(i), Show that $\chi_\rho(\sigma)$ is the number of fixed points of $\sigma \in S_n$, that is, the number of elements $k \in \{1, \cdots, n\}$ such that $\sigma(k) = k$.

(ii), Show that if $n = 3$, then $\langle \chi_\rho, \chi_\rho \rangle = 2$ and hence ρ is not irreducible.

9, Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of G of dimension d and with character χ. Show that $\ker \rho = \{g \in G \,|\, \chi(g) = d\}$. Show further that $|\chi(g)| \leq d$ for all $g \in G$, with equality only if $\rho(g) = \zeta I$, a scalar multiple of the identity, for some root of unity ζ.