Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§6 (February 27, 2009)

Maass-Selberg Relation:

$$\Lambda^T\varphi(z)=\varphi^T(z):=\begin{cases}\varphi(z) & \mathrm{Im}z<T;\\ \varphi(z)-\varphi_P(y) & \mathrm{Im}z>T.\end{cases}$$

$$\begin{aligned}\left(E^T(\cdot,s_1),\overline{E^T(\cdot,s_2)}\right)_{\mathcal{F}}&=\frac{T^{s_1+s_2-1}}{s_1+s_2-1}+\frac{\phi(s_1)T^{s_2-s_1}}{s_2-s_1}+\frac{\phi(s_2)T^{s_1-s_2}}{s_1-s_2}\\&+\frac{T^{1-s_1-s_2}}{1-s_1-s_2}\phi(s_1)\phi(s_2)\end{aligned}$$

where $s_1,s_2\in\mathbb{C}$. The poles coming from the denominations cancel: $s_1+s_2=1$, $s_1=s_2$. We have

$$E_P(y)=y^s+\phi(s)y^{1-s},$$

where

$$\phi(s)=\frac{\zeta^*(2s-1)}{\zeta^*(2s)},$$

note that $\overline{\phi(s)}=\phi(\overline{s})$, and in particular, $|\phi(s)|=1$ for $\mathrm{Re}s=\frac{1}{2}$, and hence holomorphic. Take $s_2=\overline{s_1}$, $s_1=\sigma+i\tau$, $\tau\neq 0$, $\sigma>\frac{1}{2}$,

$$\left\|\Lambda^T E(s)\right\|^2_{L^2(\mathcal{F})}=\frac{T^{2\sigma-1}}{2\sigma-1}+\frac{T^{1-2\sigma}}{1-2\sigma}|\phi(s)|^2+\frac{T^{2i\tau}}{2i\tau}\overline{\phi(s)}-\frac{T^{-2i\tau}}{2i\tau}\phi(s). \tag{1.1}$$

Recall E and ϕ has only simple pole at $\mathrm{Re}s>\frac{1}{2}$ and they are all real.

What are the residues?

(i) $\mathop{\mathrm{Res}}\limits_{s=s_0}E(s)$ is square-integrable $\in L^2(\Gamma\setminus\mathbb{H})$,

(ii)eigenfunction of Δ is $\lambda=s_0(1-s_0)$.

Proof.

$$(s-s_0)\Delta E(z;s)=(s-s_0)s(1-s)E(z;s),$$

take limit as $s\to s_0$ and we get the second assert.

$$E_P(y)=y^s+\phi(s)y^{1-s}\Rightarrow(\mathrm{Res}E(s))_P=\mathop{\mathrm{Res}}\limits_{s=s_0}\phi(s)y^{1-s_0},$$

since $\mathrm{Res}E(s)-(\mathrm{Res}E(s))_P$ is rapidly decreasing, so we have

$$\mathrm{Res}E(s)\in L^2(\mathcal{F})\leftrightarrow(\mathrm{Res}E(s))_P\in L^2(\mathcal{F}).$$

We have

$$\int_{\mathcal{F}}\left(y^{1-s_0}\right)^2\frac{\mathrm{d}x\mathrm{d}y}{y^2}=\int_0^{+\infty}y^{2(1-s_0)}\frac{\mathrm{d}y}{y^2}<+\infty,$$

where $c>0$, and $-2s_0<-1$. Hence $(\mathrm{Res}E(s))_P\in L^2(\mathcal{F})$.

Multiplying (1.1) on both sides by τ^2 and taking $\tau \to 0$, we have

$$\left\| \lim_{\tau\to 0} \Lambda^T \tau E(s_0 + i\tau) \right\|^2 = \left\| \Lambda^T \operatorname*{Res}_{s=s_0} E \right\|^2$$

$$= \frac{T^{1-2s_0}}{1-2s_0} \left| \operatorname*{Res}_{s=s_0} \phi \right|^2 + \mathrm{Res}\phi.$$

Taking $T \to \infty$, We get

$$\left\| \Lambda^T \mathrm{Res}E \right\|^2 \longrightarrow \|\mathrm{Res}E\|^2_{L^2(\Gamma\backslash\mathbb{H})} = \mathrm{Res}\phi.$$

In our case, the only pole (for $\mathrm{Re}s > \frac{1}{2}$) is at $s = 1$ and $\mathrm{Res}E = $ constant. This is true whenever Γ is a congruence subgroup, i.e.,

$$\Gamma = \mathrm{Ker}(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})).$$

In other cases, there are additional residues. Let us see why $\operatorname*{Res}_{s=1}$ is a constant. It is an eigenfunction with eigenvalue 0 and it is in $L^2(\Gamma\backslash\mathbb{H})$. Therefore it is a constant.

Finally, we get

$$c = \operatorname*{Res}_{s=1} E(z,s) = \operatorname*{Res}_{s=1} \phi(s). \quad \text{(independent of } z\text{)}$$

By (1.2)

$$c^2 \mathrm{vol}(\Gamma\backslash\mathbb{H}) = c \to c = \mathrm{vol}(\Gamma\backslash\mathbb{H})^{-1}.$$

In the case of $\Gamma = SL_2(\mathbb{Z})$, we have $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \frac{\pi}{3}$.

Remark. We know that $E(z;s)$ has a pole at $s = 1$ since $E(z;s)$ is positive and does not converge at $s = 1$.

Spectral Expansion for Δ on $L^2(\Gamma\backslash\mathbb{H})$.

On $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$, Δ is discrete, so there exists an orthonormal basis of cusp forms $\{u_j\}_{j=1}^{\infty}$ such that $(\Delta + \lambda_j)u_j = 0$ and $\varphi \in L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$, $\varphi(z) = \sum_{j=1}^{\infty} (\varphi, u_j)u_j$, $\|\varphi\|^2 = \sum |(\varphi, u_j)|^2$.

Question. What about $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})^{\perp}$?

It is the closure of $E_f = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\gamma z)$, $f \in C_c^\infty(\mathbb{R}_{>0})$. Recall

$$(E_f, \varphi)_{\Gamma\backslash\mathbb{H}} = \int_0^{+\infty} f(y)\overline{\varphi_P}(y)\frac{\mathrm{d}y}{y^2}.$$

Thus, if $(E_f, \varphi)_{\Gamma\backslash\mathbb{H}} = 0$, $\forall f \in C_c^\infty(\mathbb{R}_{>0})$, we have the right hand side $= 0$, then $\varphi_P(y) \equiv 0$, and so $\varphi \in L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$.

It remains to study the spectral theory of Δ on the space of E_f. Recall

$$E_f(z) = \int_{\mathrm{Re}s = s_0} \hat{f}(s)E(z;s)\mathrm{d}s,$$

provided that $s_0 > 1$. Also,

$$(E_f, E_g) = \int_{\mathrm{Re}s=s_0} \hat{f}(s)\left[\hat{\overline{g}}(1-s) + \phi(s)\hat{\overline{g}}(s)\right] \mathrm{d}s,$$

where

$$\hat{f}(s) = \int_{\mathbb{R}>0} f(x)x^{-s}\frac{\mathrm{d}x}{x}.$$

Shifting the contour to $\mathrm{Re}s = \frac{1}{2}$, we get

$$\int_{\mathrm{Re}s=\frac{1}{2}} \hat{f}(s)\left[\hat{\overline{g}}(1-s) + \phi(s)\hat{\overline{g}}(s)\right] \mathrm{d}s + \sum_{j=1}^{\infty} \operatorname*{Res}_{s=s_j}\phi(s)\hat{f}(s_j)\hat{\overline{g}}(s_j),$$

where $s_n, \cdots, s_m \in \mathbb{R}$ and $s_n, \cdots, s_m > \frac{1}{2}$ are the poles of ϕ.

Claim.

(i) $\hat{f}(s)\left[\hat{\overline{g}}(1-s) + \phi(s)\hat{\overline{g}}(s)\right] = \frac{1}{2}(E_f, E(s))_{\Gamma\backslash\mathbb{H}}(E(s), E(g))_{\Gamma\backslash\mathbb{H}}$;

(ii) $\hat{f}(s_j) = (E_f, \operatorname*{Res}_{s=s_j} E / \left\|\mathrm{Res}E\right\|^2)$.

Proof. 1) We have computed

$$(E_f, E(s)) = \hat{f}(1-\bar{s}) + \bar{\phi}(s)\hat{f}(\bar{s}), \tag{1.2}$$

and

$$(E(s), E_g) = \hat{\overline{g}}(1-s) + \phi(s)\hat{\overline{g}}(s).$$

For $\mathrm{Re}s = \frac{1}{2}$, $1 - \overline{s} = s$, $|\phi(s)| = 1$, combine the last two identities and we get (i).

2) Taking $\operatorname*{Res}_{s=s_j}$ of (1.3), we have

$$(E_f, \mathrm{Res}E(s)) = \mathrm{Res}\phi\hat{f}(s_j),$$

using $\left\|\mathrm{Res}E\right\|^2 = \mathrm{Res}\phi(s_j)$, we get (ii).

Theorem. Let $\{u_j\}_{j=0}^{\infty}$ be an orthonormal basis of eigenfunctions of Δ, $(\Delta + \lambda_j)u_j = 0$, then for any $\varphi \in L^2(\Gamma\backslash\mathbb{H})$, we have

$$\varphi = \sum_{j=0}^{\infty}(\varphi, u_j)u_j + \frac{1}{4\pi}\int_{\mathrm{Re}=\frac{1}{2}} (\varphi, E(\cdot, s))\, E(\cdot, s)\mathrm{d}s.$$

where

$$\sum_{j=0}^{\infty}(\varphi, u_j)u_j$$

is the discrete part which contains the cuspidal forms and the residues of Eisenstein series at $s > \frac{1}{2}$, and

$$\frac{1}{4\pi}\int_{\mathrm{Res}=\frac{1}{2}} (\varphi, E(\cdot, s))\, E(\cdot, s)ds$$

is the continuous part which converges if ϕ is "nice".

$$-\Delta\varphi=\sum_{j=0}^{\infty}\lambda_j(\varphi,u_j)u_j+\frac{1}{4\pi}\int_{\mathrm{Re}s=\frac{1}{2}}s(1-s)\left(\varphi,E(\cdot,s)\right)E(\cdot,s)\mathrm{d}s.$$

We can write $0=\lambda_0\le\lambda_1\le\lambda_2\le\cdots$ and

$$u_0=\frac{1}{(\mathrm{vol}(\Gamma\backslash\mathbb{H}))^{\frac{1}{2}}}$$

which is a constant. For $\Gamma=SL_2(\mathbb{Z})$, $u_j(j>0)$ are cuspidal forms.
Special case: If k is a point-pair invariant function, recall

$$K(z,w)=\sum_{\gamma\in\Gamma}k(z,\gamma_\omega),$$

In order to expand $K(\cdot,\omega)$, we write $\lambda_j=s_j(1-s_j)$, and $s_j=\frac{1}{2}+\mathrm{i}t_j$. we know that

$$\int K(z,w)\overline{u_j(z)}d\mu(z)=h(t_j)\overline{u_j(\omega)},$$

where h is the Selberg transform of k. Finally, we get

$$K(z,\omega)=\sum_{j=0}^{\infty}h(t_j)u_j(z)\overline{u_j(\omega)}+\frac{1}{4\pi}\int_{\substack{\mathrm{Re}s=\frac{1}{2}\\ s=\frac{1}{2}+\mathrm{i}r}}h(r)E(z,s)\overline{E(\omega,s)}ds,$$

which is the spectral expansion of the automorphic kernel.

Another way to work with the spectral expansion is Eisenstein transform

$$f\mapsto E_f=\int_0^{\infty}f(r)E(z,\frac{1}{2}+\mathrm{i}r)\mathrm{d}r,$$

where $f\in C(\mathbb{R}_{>0})$, and

$$\|E_f\|^2_{L^2(\Gamma\backslash\mathbb{H})}=\|f\|_{L^2(\mathbb{R}_{>0})},$$

$$\varphi=\sum(\varphi,u_j)u_j+\frac{1}{4\pi}\int(\varphi,E)Eds,$$

$$(\varphi_1,\varphi_2)_{\Gamma\backslash\mathbb{H}}=\sum_j(\varphi_1,u_j)(u_j,\varphi_2)+\frac{1}{4\pi}\int_{\mathrm{Re}s=\frac{1}{2}}(\varphi_1,E(s))(E(s),\varphi_2)\mathrm{d}s.$$

Continuous part of the spectrum of Δ on $L^2(\Gamma\backslash\mathbb{H})$ is well-understood. and it is $-[\frac{1}{4},+\infty)$,

$$\Delta E(\frac{1}{2}+\mathrm{i}t)=-\left|\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2E(\frac{1}{2}+it).$$

What about discrete part?
Question.

i) $\sharp\{j:\lambda_j\le T\}\sim ?$

ii) What can λ_1 be? λ_1 controls the error term in various problems, such as hyperbolic lattice point counting problem.

Selberg eigenvalue conjecture.

$$\lambda_1(\Gamma_N\backslash\mathbb{H}) \geq \frac{1}{4},$$

i.e, cuspidal spectrum $\subseteq$ continuous spectrum, where $\Gamma_N = \mathrm{Ker}(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}))$. One interesting thing is that forms with eigenvalue $\frac{1}{4}$ are expected to come from arithmetic (Galois representation).

Prop. (Selberg)

$$\lambda_1\left(SL_2(\mathbb{Z})\backslash\mathbb{H}\right) \geq \frac{3}{2}\pi^2.$$

Proof. Let u_1 be a cusp form with $\|u_1\| = 1$, $u_1 = \sum_{n\neq 0} c_n(y)e(nx)$,

$$\lambda_1 = -(\Delta u_1, u_1)_{\Gamma\backslash\mathbb{H}} = \int_{\mathcal{F}} |y\nabla u_1(z)|^2 \,\mathrm{d}\mu(z),$$

where

$$(\Delta f, f) = \|\nabla f\|^2.$$

Let

$$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

we have

$$\mathcal{F}\bigcup\omega\mathcal{F} \supseteq \{z\in\mathbb{H}, |\mathrm{Re}z| < \frac{1}{2}, \mathrm{Im}z > \frac{\sqrt{3}}{2}\}.$$

Then

$$\begin{aligned}
2\lambda &= \int_{\mathcal{F}\bigcup\omega\mathcal{F}} |y\nabla u_1(z)|^2 \,\mathrm{d}\mu(z) \geq \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_0^1 |y\nabla u_1(z)|^2 \,\frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&\geq 3\pi^2 \int_{\frac{\sqrt{3}}{2}}^{+\infty} \sum_{n\neq 0} |c_n(y)|^2 \frac{\mathrm{d}y}{y^2} = 3\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_0^1 |u_1(x+iy)|^2 \mathrm{d}x\frac{\mathrm{d}y}{y^2} \\
&\geq 3\pi^2 \int_{\mathcal{F}} |u_1(z)|^2 \mathrm{d}\mu(z) = 3\pi^2.
\end{aligned}$$

This argument also gives $\lambda_1 > \frac{1}{4}$ for other Γ_N for small N. But there are examples of $\Gamma's$ (non-congruence), where $\lambda_1 < \frac{1}{4}$ (in fact, λ_1 can be as arbitrarily close to 0).

Small application of Eisenstein series. Recall

$$E(z,s) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{Q_2((m,n))^s}$$

has an arithmetic meaning for special z (CM points, $z^2 + rz + q = 0,\ r, q\in\mathbb{Z}$).

Suppose d is a square free number satisfying $d \equiv 1 (\bmod 4)$, $d < 0$. Denote V as the set of solutions of quadratic equations of discriminant d, and let $\Lambda_d = \Gamma \backslash V$, then we have

$$\sum_{z \in \Lambda_d} E(z, s) = \zeta_{K_d}(s) = \zeta(s) L(s, \chi_d),$$

where $K_d = \mathbb{Q}(\sqrt{d})$ is an imaginary quadratic number field. In fact,

$$\begin{aligned} \Lambda_d &\Leftrightarrow \text{ ideal classes of } \mathbb{Q}(\sqrt{d}) \\ &\Leftrightarrow \sim \backslash \{ \text{ binary quadratic forms of discriminat d} \}. \end{aligned}$$

Consider

$$\sum_{z \in \Lambda_d} E(z; s) = \zeta_{K_d}(s) = \zeta(s) L(x, \chi_d).$$

Assume RH is FALSE, i.e., there exists s_0, $\mathrm{Re} s_0 > \frac{1}{2}$ such that $\zeta(s_0) = 0$, then we have

$$\sum_{z \in \Lambda_d} E(z; s_0) = 0.$$

We will show that $h(d)$ can not be 1 for d unbounded, i.e., there are only finitely many imaginary quadratic fields with class number 1.

Proof. $E(s_0) - E_P(s_0)$ is rapidly decreasing as $y \to \infty$. If $h(d) = 1$, $\Lambda_d = \{\frac{1+\sqrt{d}}{2}\}$, then $E(\lambda_d) = 0$.

On the other hand,

$$E(\lambda_d) = |d|^{s_0} + \phi(s_0)|d|^{1-s_0} + O(|d|^{-N})$$

implies that

$$E(\lambda_d) \neq 0$$

for d large, because $\mathrm{Re(s_0)} > \frac{1}{2}$ and $|d^{s_0}| = d^{\mathrm{Re} s_0}$.

Deuning (1920s)

This argument was pushed by Siegel to show that $h(d) \sim \sqrt{|d|}$ ineffectively (clear under GRH).

Goldfeld-Gross-Zagier (1980s) roughly proved that $h(d) \gg \log d$ effectively, the upper bound $h(d) \ll \sqrt{|d|} \log d$ is trivial.

Main impact. Existence of high order zeros at $\frac{1}{2}$ for L-functions.