Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§2 Geometry and Analysis on the Hyperbolic Plane (February 19, 2009)

The upper-half plane. As a model of the hyperbolic plane we will use the upper half plane:

$$\mathbb{H} = \{z = x + iy, y > 0\}.$$

$\mathbb{H}$ is a Riemannian manifold with the metric derived from the Poincaré differential, $\mathrm{d}s = \frac{|\mathrm{d}z|}{y}$, it also can be written as $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. In the hyperbolic plane, the angle is the same as in the Euclidean plane and the distance function on $\mathbb{H}$ is given explicitly by

$$\rho(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}.$$

For example, $\rho(it, is) = |\log \frac{t}{s}|$. We have

$$\cosh \rho(z, w) = 1 + 2u(z, w),$$

where

$$u(z, w) = \frac{|z - w|^2}{4\mathrm{Im}z\mathrm{Im}w}.$$

The group $G = \mathrm{SL}_2(\mathbb{R})$ acts by isometries on $\mathbb{H}$ via Möbius transformations:

$$gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}, \quad \forall g \in G.$$

By the definition we can easily get $\mathrm{Im}gz = \dfrac{\mathrm{Im}z}{|cz + d|^2}$, $gz - gw = \dfrac{z - w}{(cz + d)(cw + d)}$ and $\dfrac{\mathrm{d}}{\mathrm{d}z}gz = (cz + d)^{-2}$. Moreover, we have $\dfrac{|\mathrm{d}gz|}{\mathrm{Im}gz} = \dfrac{|\mathrm{d}z|}{\mathrm{Im}z}$. Consider

$$C_g = \{z \in \mathbb{H} : |cz + d| = 1\}.$$

If $c \neq 0$, this is a semi-circle centered at $-\frac{d}{c}$ of radius $|c|^{-1}$. Therefore, g acts on C_g as an euclidean isometry.

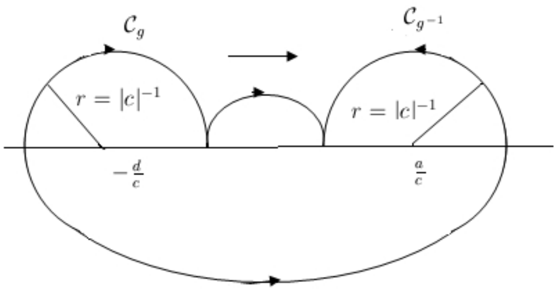

Figure 1

Matrix $-I$ acts as the identity. Then in fact, $\mathrm{PSL}_2(\mathbb{R}) = G/\{\pm I\}$ of all Möbius transformations, acts on the whole compactified complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by complex automorphism. $\widehat{\mathbb{C}}$ is a Riemann sphere and it splits into three G-invariant subspaces, namely $\mathbb{H}$, $\overline{\mathbb{H}}$ and $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By convention, a straight line is a circle with an infinite radius. Then any Möbius transformation g takes circles to circles, $C_1 \overset{g}{\rightsquigarrow} C_2$, but the center of $C_1 \not\twoheadrightarrow$ the center of C_2. G acts transitively on $\mathbb{H}$. It acts simply transitively on $\{(z, w) \in \mathbb{H} \times \mathbb{H} : \rho(z, w) = a\}$ for any $a > 0$. Also G acts transitively on geodesics (e.g., $i\mathbb{R}_{>0}$ is a geodesic).

Theorem. All geodesic semi-circles and straight lines are orthogonal to $\widehat{\mathbb{R}}$.

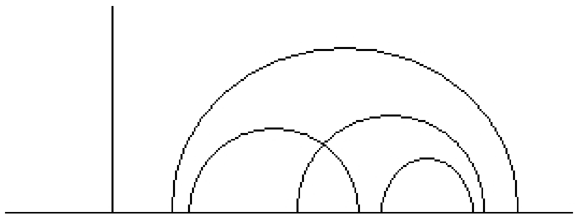

FIGURE 2

Given a geodesic C and a point p outside C, then there are precisely two geodesics passing through p which are tangent to C.

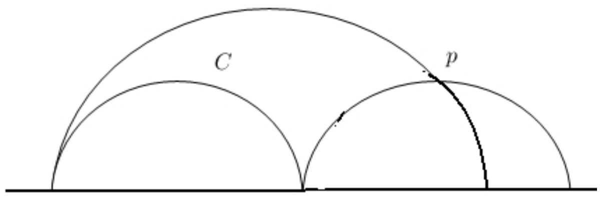

FIGURE 3

The hyperbolic circles $\{z \in \mathbb{C} : \rho(z, z_0) = r\}$ are also euclidean circles. If $\rho(z, i) = r$, we have $|z - i\cosh r| = \sinh r$, where $\cosh r = \frac{e^r + e^{-r}}{2}$ and $\sinh r = \frac{e^r - e^{-r}}{2}$.

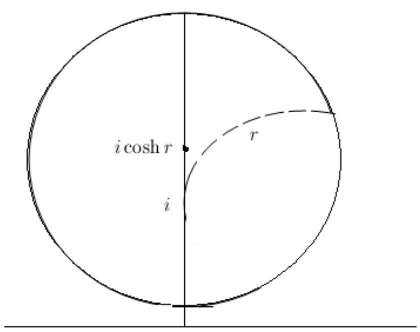

FIGURE 4

It is easy to see that the Möbius transformations are isometries of the hyperbolic plane.

The Classification of Motions. Notice that any $g \in G$ is conjugate to:
$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ -parabolic, $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$-hyperbolic or $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$-elliptic.

Another way to think it out this classification is according to the number of fixed points on $\widehat{\mathbb{R}}$. Because $gz = z$ always has two solutions in $\widehat{\mathbb{C}}$, so there are three cases:

- g has one fixed point in $\mathbb{H}$, (elliptic case)
- g has two distinct fixed points on $\widehat{\mathbb{R}}$, (hyperbolic case)
- g has one double fixed point on $\widehat{\mathbb{R}}$. (parabolic case)

(i) If g is elliptic, suppose z_0 is the fixed point of g, then g stabilizes $\rho(z_0, z) = r$ and g moves points along circles around z_0.(rotation)

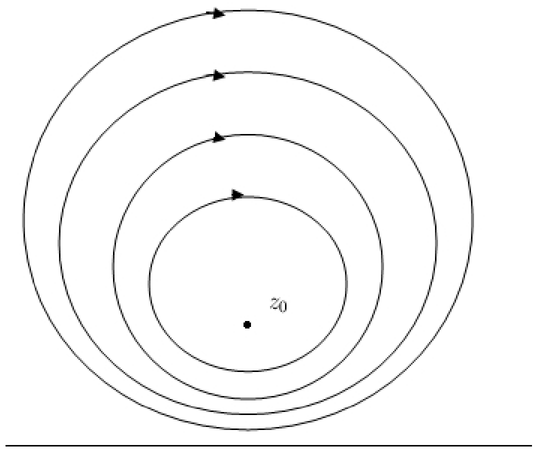

FIGURE 5. elliptic

(ii) If g is hyperbolic, there are two fixed points $x_1, x_2 \in \hat{\mathbb{R}}$ and g moves points along hyper-cycles in $\mathbb{H}$ (segments of circles in $\mathbb{H}$ passing through x_1, x_2). Of the two fixed points, one is repelling and the other is attracting.

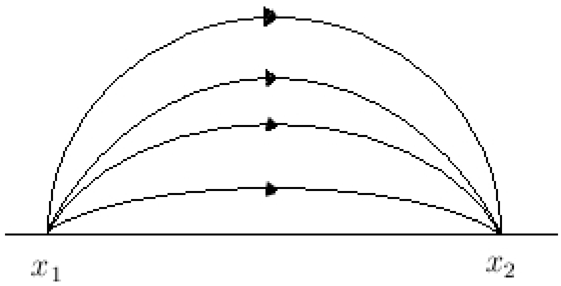

FIGURE 6. hyperbolic

(iii) If g is parabolic, there is a double fixed point x_0 and g moves points along horocycles circles in $\mathbb{H}$ which are tangent to $\hat{\mathbb{R}}$ at x_0.

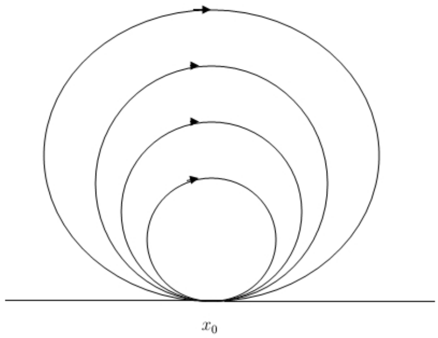

FIGURE 7. parabolic

Note that only elliptic ones g can be of finite order.

Let area measure $\mathrm{d}\mu z = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$ and one can easily check it is G-invariant, where $G = SL_2(\mathbb{R})$.

EX. Area of a hyperbolic disc of radius r is $4\pi(\sinh\frac{r}{2})^2$ and the Euclidean area is $\pi(\sinh r)^2$. The circumference (length of a circle) of radius r is $2\pi\sinh r$ which is the same as the Euclidean case.

For a disc of radius r, most of the area is near the boundary.

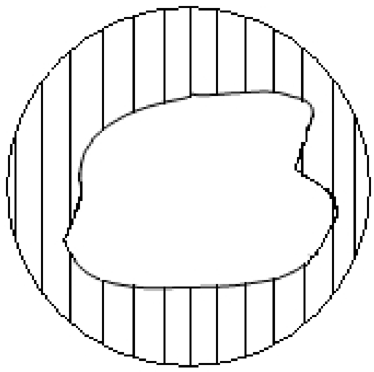

FIGURE 8

There is a universal inequality between the area and the boundary length of a domain in a Reimannian surface which is called the isoperimetric inequality; it asserts that

$$4\pi A - KA^2 \leq L^2,$$

where A is the area, L is the length of the boundary, and K is the curvature.

In the Euclidean plane we have $K = 0$ and $A \leq L^2$. On the other hand, in the hyperbolic plane we have $K = -1$ and $A \leq L$. In fact, we can have $A \sim L$.

The same classification can be also described in terms of trace, namely if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm I$, then

(i) g is elliptic $\iff |\mathrm{tr} g| < 2$,
(ii) g is hyperbolic $\iff |\mathrm{tr} g| > 2$,
(iii) g is parabolic $\iff |\mathrm{tr} g| = 2$.

$\mathbb{H}$ as a Homogeneous Space.

Let $\mathbb{H} \cong G/K$, and

$$K = \mathrm{Stab}(i) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Under this identification, G acts by left regular action. Suppose the rectangular coordinates $z = x + iy$, we have

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} i = x + iy = z.$$

Iwasawa Decomposition $G = NAK$**.** For GL_n, Gram-Schmidt process $\Rightarrow \forall g \in GL_n(\mathbb{R})$ can be written uniquely as

$$g = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} k,$$

where $k \in O(n, \mathbb{R})$, $a_1, \ldots, a_n > 0$.

$GL_n = G = NAK$, where $A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, a_1, \ldots, a_n > 0 \right\}$, $K = O(n, \mathbb{R})$ which is the maximal compact subgroup and $N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $NA = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$. A normalizes N.

The map

$$N \times A \times K \to G,\ n, a, k \mapsto nak$$

is the diffeomorphism of manifolds.

In our case,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is the unique decomposition.

Consider the measure$\frac{\mathrm{d}x\mathrm{d}y}{y^2}$ on $G/K \cong \mathbb{H}$. Any locally compact group has a left invariant Haar measure $\mathrm{d}g$, say, which means that

$$\int f(hg)\mathrm{d}g = \int f(g)\mathrm{d}g,$$

for any test function integrable on G. The left invariant measure is unique up to a constant; therefore,

$$\int f(gh)\mathrm{d}g = \delta(h) \int f(g)\mathrm{d}g, \qquad \delta(h) > 0,$$

where $\delta : G \to \mathbb{R}_+$ is a homomorphism of groups. G is called unimodular if $\delta \equiv 1$ and $\mathrm{d}g$ is right invariant. For example, any abelian and compact group are unimodular, and

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$

is not unimodular. It is easy to check that

$$\delta \begin{pmatrix} a^{1/2} & * \\ 0 & a^{-1/2} \end{pmatrix} = a^{-1}.$$

As above, for $G = GL_2$, the Iwasawa decomposition is

$$G = NAK,$$

where

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R},$$

$$A = \left\{ a(y) = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} : y > 0 \right\} \cong \mathbb{R},$$

and

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong SO(2).$$

The corresponding invariant measures on N, A, K are given by

$$\mathrm{d}n(x) = \mathrm{d}x,\ \mathrm{d}a(y) = y^{-1}\mathrm{d}y,\ \mathrm{d}k(\theta) = (2\pi)^{-1}\mathrm{d}\theta,$$

where $\mathrm{d}x, \mathrm{d}y, \mathrm{d}\theta$ are the Lebesgue measures. Since K is compact, we could normalize the measure on K to have $\int_K \mathrm{d}k = 1$.

Consider the following integral

$$\int_G f(g)\mathrm{d}g = \int_A \int_N \int_K f(ank)\mathrm{d}a\mathrm{d}n\mathrm{d}k,$$

we have known that $G/K \cong \mathbb{H}$ and $G/K = NA = AN = P$. Notice that A and N are abelian, yet the following commutativity relation holds:

$$a(y)n(x) = n(xy)a(y).$$

In the following let us define a measure $\mathrm{d}p$ on $P = AN$ by requiring that

$$\int_P f(p)\mathrm{d}p = \int_A \int_N f(an)\mathrm{d}a\mathrm{d}n,$$

i.e., $\mathrm{d}p = y^{-1}\mathrm{d}x\mathrm{d}y$ if $p = a(y)n(x)$. We can show that $\mathrm{d}p$ is left invariant. Furthermore, by Fubini's theorem we derive the relation

$$\begin{aligned}\int_A\int_N f(an)\mathrm{d}a\mathrm{d}n &= \int_{\mathbb{R}}\int_{\mathbb{R}^+} f(a(y)n(x))y^{-1}\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^+} f(n(xy)a(y))y^{-1}\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^+} f(n(x)a(y))y^{-2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

This shows that the modular function of P is equal to $\delta(p) = y^{-1}$ if $p = n(x)a(y)$. Hence the left invariant measure on P is equal to $\delta(p)\mathrm{d}p = y^{-2}\mathrm{d}x\mathrm{d}y$, which is just the Riemannian measure on $\mathbb{H}$.

Cartan Decomposition $G = KAK$. For GL_n, given $g \in GL_n(Q)$, gg^t is positive definite $\Rightarrow \exists k \in K$ satisfying $gg^t = ka^2k^{-1}$, where $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, $a_1 \geq \cdots a_n > 0 \Rightarrow \exists k' \in K,\ g = kak'$. We shall write any $g \in PSL_2(\mathbb{R})/K$ uniquely as $g = k(\varphi)a(e^{-r})k(\theta)$, $r \geq 0$. The pair (r, ϕ) is called the geodesic polar coordinate of the point z, $\rho(gi, i) = \rho(k(\varphi)a(e^{-r})k(\theta)) = \rho(e^{-r}i, i) = r$.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} * & * \\ -y^{\frac{1}{2}}\sin\theta & y^{\frac{1}{2}}\cos\theta \end{pmatrix}.$$

$(y, \theta + \frac{\pi}{2})$ are the polar coordinates of (c, d). The length element and the measure are expressed as follows:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (2\sinh r)^2\mathrm{d}\varphi^2, \quad \mathrm{d}\mu z = (2\sinh r)\mathrm{d}r\mathrm{d}\varphi,$$

where $\cosh r = 1 + 2u$ as above, we have

$$\mathrm{d}\mu z = 4\mathrm{d}u\mathrm{d}\varphi.$$

The Laplace Operator. In rectangular coordinates,

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

and in geodesic polar coordinates (r, φ), the Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r}\frac{\partial}{\partial r} + \frac{1}{(2\sinh r)^2}\frac{\partial^2}{\partial \varphi^2}.$$

Denote by T_g the following operator:

$$T_g f(z) = f(gz).$$

Using the definition, we have

$$\Delta T_g = T_g\Delta, \quad \forall g \in G.$$

EX. The ring of the differential operators on $\mathbb{H}$ which commutes with G is $\mathbb{C}[\Delta]$.
Eigenfunctions of Δ. A function $f : \mathbb{H} \to \mathbb{C}$ with continuous partial derivatives of order 2 is an eigenfunction of Δ with eigenvalue $\lambda \in \mathbb{C}$ if

$$(\Delta + \lambda) f = 0.$$

Question: How to construct? Obviously, y^s is an eigenfunction with eigenvalue $\lambda = s(1-s)$. One basic trick: if f is an eigenfunction, then $T_g f$ is also an eigenfunction, moreover, $\int_G T_g f \mathrm{d}\mu(g)$ is also an eigenfunction for any measure μ on G.

Suppose that we want an eigenfunction of the form $e(x)F(2\pi y)$, by separation of variables, F satisfies

$$F''(y) + (\lambda y^{-2} - 1)F(y) = 0, \quad \lambda = s(1-s).$$

The basis of solutions are $y^{-\frac{1}{2}} K_{s-\frac{1}{2}}(y),\ y^{\frac{1}{2}} I_{s-\frac{1}{2}}(y)$ which are asymptotic to $e^{-y},\ e^{y}$ as $y \to \infty$, respectively. Whittaker function is defined as

$$W_s(z) = 2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi y) e(x).$$

Alternatively, we want $f(nz) = \chi(n) f(z),\ \chi(n) = e(x)$. Starting with $(\mathrm{Im} z)^s$ and averaging, we obtain

$$\begin{aligned}
\int_N \bar{\chi}(n) (\mathrm{Im} wnz)^s \mathrm{d}u &= \int_{\mathbb{R}} e(t) \Big(\mathrm{Im}(\frac{-1}{z-t})\Big)^s \mathrm{d}t \\
&= e(x) y^{1-s} \int_{-\infty}^{+\infty} (1+t^2)^{-s} e(ty) \mathrm{d}t \\
&= \frac{\pi^s}{\Gamma(s)} W_s(z).
\end{aligned}$$

Here the involution $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ was inserted to buy the absolute convergence, at least if $\mathrm{Re} s > \frac{1}{2}$.