EXERCISE 5

1. a) Show that the holomorphy of the Eisenstein series on $\mathrm{Re}s = \frac{1}{2}$ implies PNT : $\zeta(1+it) \neq 0, \ \forall t \in \mathbb{R}$.

b) Show that

$$\int_1^\infty \Big|\frac{K_{it}(2\pi y)}{\Gamma(\frac{1}{2}+it)}\Big|^2 \frac{dy}{y} \gg \frac{1}{t}$$

by using the asymptotic

$$K_{it}(y) \sim \frac{e^{-\frac{\pi}{4}}\sqrt{2\pi}}{\sqrt[4]{t^2-y^2}} \sin(\frac{\pi}{4} + th(\frac{y}{t}))$$

for $y < \frac{t}{4}$ where h is a smooth function.

c) By integrating over the fundamental domain of $\mathrm{SL}_2(z)$ and using

$$\int_0^1 |f(x+iy)|^2 dx \geq | \int_0^1 f(x+iy)e(x)dx|^2$$

deduce that

$$|\zeta(1+it) \gg \frac{1}{t(\log t)^2}|.$$

(For more information, cf. P. Sarnak, Shalika 60th Birthday in www.math.princeton.edu/Sarnak)

2. Show that the estimate

$$\sum_{P:\ p\leq X} \log p = X + O(X^\theta), \ \theta < \frac{1}{2},$$

(where P ranges over primitive hyperbolic conj. classes and $p = NP$) is equivalent to

$$\sharp\{P:\ p \leq X\} = LiX + O(X^\theta)$$

where $LiX = \int_2^X \frac{dt}{\log t}$.

3. Show that there is a natural correspondence between primitive hyperbolic conj. classes of $\mathrm{SL}_2(z)$ and equivalence classes of primitive binary quadratic forms of discriminant $d > 0, \ d \equiv 0, 1 \ (4)$, and d is not a square.