Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§1 Introduction to automorphic forms (February 17, 2009)

What is an automorphic form? Automorphic forms are generalization of periodic functions, i.e.,

$$f \ : \ \mathbb{R} \to \mathbb{C}, \ f(x+1) = f(x).$$

Two important things:

(i) We have an explicit basis of periodic functions, i.e., $e(mz), \ m \in \mathbb{Z}$, where $e(z) = e^{2\pi i z}$.

(ii) We can expand any periodic function explicitly in terms of $e(mz)$, i.e.,

$$f(z) = \sum a_f(m) e(mz), \ a_f(m) = \big(f, e(mz)\big).$$

Poisson Summation Formula (PSF) For any Schwarz function f on $\mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where

$$\hat{f}(y) = \int f(x) e(xy) \mathrm{d}x.$$

A typical application of PSF : Suppose $f \in L^1(\mathbb{R})$ and $f^{(n)} \in L^1(\mathbb{R})$ (e.g. $f(x) = \frac{1}{1+x^2}$), then for any $k > 0$ we have

$$\Big| \frac{1}{X} \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{X}\Big) - \int_{\mathbb{R}} f(x)\mathrm{d}x \Big| = O_k(X^{-k}).$$

Note that $\int_{\mathbb{R}} f(x)\mathrm{d}x = \hat{f}(0)$.

Suppose that Γ is a discrete group, which acts discontinuously on a locally compact space X, i.e. $\Gamma \times X \to X$ is a proper map. $\forall \ C \subseteq X$ is compact, the set

$$\{\gamma \in \Gamma \ : \ \gamma C \cap C \neq \emptyset\}$$

is finite. An **automorphic function** on X is a $\Gamma-$invariant function on X, i.e., $f \ : \ X \to C$, $f(\gamma x) = f(x)$, $\forall \ \gamma \in \Gamma$, $x \in X$.

A particular case : X is a symmetric space (certain Riemannian manifold). 3 kinds according to sectional curvature.

(I) Euclidian space $\mathbb{R}^n$ (flat).

(II) S^n, sphere (compact type).

(III) G/K, G is a semi-simple algebraic lie group, K is a maximal compact subgroup of G (K is unique up to conjugation).

EX. (I) $G = SL_n(\mathbb{R})$, $K = SO_n(\mathbb{R})$, then G/K is a space of positive definite $n-$forms.

(II) $G = SL_n(\mathbb{C})$, $K = SU(n)$, then G/K is a space of positive definite Hermitian $n-$forms.
(III) $G = SO(n,1)$, $K = SO(n)$, then G/K is a hyperbolic n-space ($\mathbb{H}^n$).

Let $X = G/K$. Mostly, we work with $SL_2(\mathbb{R})/SO(2) = \mathbb{H}^2$, which is a two dimensional hyperbolic space, i.e., the upper half plane $\{z \in \mathbb{C} \;:\; z = x + iy,\; y > 0\}$. Let $G = SL_2(\mathbb{R})$ act on X by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Let $K = SO(2) = \mathrm{Stab}(i)$ and $\Gamma \subseteq G$ be discrete. G acts on $X \rightsquigarrow \Gamma$ acts on X. Γ acts discontinuously on X. Mostly, Γ will be of finite co-volume in G ($X = G/K$ is a Riemannian metric, $T_\rho(\frac{G}{K}) = \mathfrak{g}/\mathfrak{k} = \mathrm{Lie}(G)/\mathrm{Lie}(K)$. In fact, let $[,]$ be the Killing form on $\mathrm{Lie}(G)$. $\mathfrak{k}$ is a maximal subspace on which $[,]$ is negative definite. Then the Killing form on the orthogonal complement of $\mathrm{Lie}(K)$ in $\mathrm{Lie}(G)$ is positive definite. It gives rise to a Riemannian metric on X). That is $\mathrm{vol}(\Gamma\backslash X) < \infty$. Volume form on $X \Rightarrow$ volume form on $\Gamma\backslash X$.

Many important cases of such Γ.

Def. A lattice Γ of G is a discrete subgroup of G of finite co-volume, i.e.,

$$\mathrm{vol}(\Gamma\backslash X) < \infty.$$

It is called uniform if $\Gamma\backslash X$ is compact $\Leftrightarrow \Gamma\backslash G$ is compact.

EX. (1) Let $G = SL_n(\mathbb{R})$, $\Gamma = SL_n(\mathbb{Z})$. One can show that Γ is a lattice (part of reduction theory, Hermite Minkowski, Borel, Harish-Chandra).
(2) For $G = SL_2(\mathbb{R})$, $X = \mathbb{H}$. There are many examples of uniform lattices. Any compact Riemann Surface with genus > 1 can be uniformized by $\mathbb{H} \Longrightarrow M = \Gamma\backslash\mathbb{H}$, where Γ is a uniform lattice of $SL_2(\mathbb{R})$. That is, there are continuous families of uniform lattices of $SL_2(\mathbb{R})$. Similarly, there are continuous families of non-uniform lattices.

For other G, the situation is very different. If G has rank > 1 ($SL_m,\; m \geq 3$), more or less, all lattices Γ are obtained as $G(\mathbb{Z})$ (Margulis)(i.e. they are arithmetic. The notion of arithmeticity is somewhat subtle. It will not be defined here).

Back to $SL_2(\mathbb{R})$ case, we have many lattices Γ's. Some are more special than others. Denote

$$\mathrm{Comm}(\Gamma) = \{g \in G \;:\; g\Gamma g^{-1} \cap \Gamma \text{ is of finite index in } \Gamma\}$$

which is a subgroup of G. Obviously, $\Gamma \subseteq \mathrm{Comm}(\Gamma)$.

Theorem (Margulis) $[\mathrm{Comm}(\Gamma) : \Gamma] = \infty \Leftrightarrow \Gamma$ is arithmetic.

Example. $\Gamma = SL_2(\mathbb{Z})$ or a finite index subgroup of $SL_2(\mathbb{Z})$. " $\Leftarrow$ " is easy for Γ, because $\mathrm{Comm}(\Gamma) = SL_2(\mathbb{Q})$. This property of Γ is very important because it gives rise to additional symmetries (Hecke operators). We consider functions on $\Gamma\backslash\mathbb{H}$. This does not include modular forms, because of the weight factor.

We can refer to references [1], [2], [3], [4], [5], [6].

Basic goal: spectral decomposition (mostly for $G = SL_2(\mathbb{R})$)

First step : study G/K (Harish-Chandra, Helgason, Gangolli).

Weyl law : Let (M, g) be a compact Riemannian manifold of dimension d. Δ-Laplace operator (2^{nd} order differential operator) which is an unbounded non-positive self-adjoint on $\mathrm{L}^2(M, \mathrm{d}\mu)$ with pure point spectrum.

$$\Delta\varphi_j + \lambda_j^2\varphi_j = 0$$

where $\{\varphi_j\}$ is an orthonormal basis in $\mathrm{L}^2(M)$ and $0 = \lambda_0 < \lambda_1 \leq \cdots$. We have

$$\sharp\{j \ : \lambda_j \leq T\} = C_d \mathrm{vol}(M)T^d + O(T^{d-1}),$$

where C_d depends only on d.

EX. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Then $\varphi_{n,m} = e(nx)e(my)$ is an eigenfunction and the eigenvalue $\lambda_{n,m} = n^2 + m^2$. By Weyl law we get

$$\sharp\big\{(n,m) \ : n^2 + m^2 \leq R^2\big\} = \pi R^2 + O(R).$$

Simplest bound for Gauss circle problem.

The Weyl law holds for $\Gamma\backslash\mathbb{H}$, where Γ is a congruence subgroup, e.g. $\Gamma = \Gamma_N$, where

$$\Gamma_N = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})).$$

Remark. Not clear how to write down a single non-constant function in $\mathrm{L}^2\big(SL_2(\mathbb{Z})\backslash\mathbb{H}\big)$, which is an eigenfunction of Δ.

To show that Selberg invented the trace formula which is a broad generalization of the Poisson Summation Formula.

Other applications of spectral theoretical methods.

Lattice point counting problem. Let (X, d) be a metric space and Γ act discontinuously on it. Given $x_0 \in X$, we have $\{\gamma \in \Gamma \ : \ \mathrm{d}(\gamma x_0, x_0) < R\}$ is finite.

What is the asymptotic behavior? (e.g. $\Gamma = \mathbb{Z}^2$, $X = \mathbb{R}^2$, $x_0 = 0$, Gauss circle problem.)

For $\mathbb{H}$, let $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ and it is $G-$invariant, where $G = SL_2(\mathbb{R})$. Let $\mathrm{d}\mu = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$ stand for the measure, $\Delta = y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$ and $\Gamma = SL_2(\mathbb{Z})$. Then we have

$$\sharp\big\{(a,b,c,d) \ : ad - bc = 1, \ a^2 + b^2 + c^2 + d^2 \leq R\big\} = 3R + O(R^{\frac{2}{3}}).$$

This is more difficult since in the hyperbolic geometry, the length of the boundary of a circle is proportional to the area !

An analogue of the prime number theorem for closed geodesics.

Theorem (Selberg). For $\Gamma = SL_2(\mathbb{Z})$, we have

$$\sum_{\substack{r \text{ closed} \\ \text{geodesic in } \Gamma\backslash\mathbb{H} \\ l(r)\leq \log X}} l(r) = X + O(X^{\frac{3}{4}}).$$

(closely related to fundamental units of real quadratic fields.)

M. Kac : Can you hear the shape of a drum? Let M be a Riemannian surface, $\triangle$ be Laplace operator and $0 = \lambda_0 < \lambda_1 \leq \cdots$. Do these eigenvalues determine M ?

No, we have $\Gamma_1\backslash\mathbb{H} \neq \Gamma_2\backslash\mathbb{H}$ with same λ_j (Jacquet-Langlands correspondence, special case of Langlands functoriality)

Smallest eigenvalue problem. Let $0 = \lambda_0 < \lambda_1 \leq \cdots$ be eigenvalues, what is λ_1?

Selberg's $\frac{1}{4}$ conjecture : If Γ is congruence subgroup ($\Gamma = \Gamma_N$), then $\lambda_1 \geq \frac{1}{4}$.

Selberg showed $\lambda_1 \geq \frac{3}{16}$. Best known bound is $\lambda_1 \geq \frac{1}{4} - (\frac{7}{64})^2$. To prove these results, one needs to consider higher rank symmetric spaces (Kim-Sarnak, Shahidi, need to use E_8). The Selberg's $\frac{1}{4}$ conjecture would follow from general Langlands functoriality.

References

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