Modes of Convergence

Notations:

u.c.–*uniform convergence.* **c.**– *pointwise convergence*

a.e.c.–*almost everywhere convergence*

n.u.c.–*nearly (almost) uniform convergence*

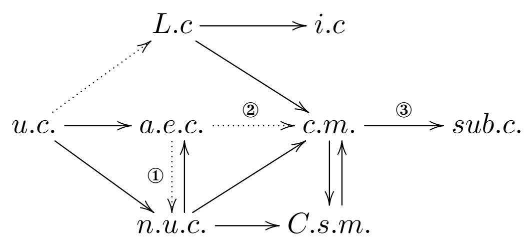
c.m.–*convergence in measure(probability):* $f_n(x) \overset{m}{\to} f(x)$.

sub.c.–*subsequence convergence :,* $f_{n_j}(x) \to f(x)$.

C.s.m.–*Cauchy sequence in measure:,* $mE[|f_m(x) - f_n(x)| > \delta] \to 0$.

L.c.–L^1 *norm convergence:* $f_n \in L(E), \int_E |f_n(x) - f(x)|\, dx \to 0$.

i.c.–*integral convergence:* $\int_E f_n(x)\, dx \to \int_E f(x)\, dx$.



Remarks:

1, *A dashed line means that convergence in the mode under* $mE < \infty$.

2, ①: *Egorov Theorem*; ②: *Lebesgue Theorem*; ③: *Riesz Theorem.*

3, *Monotone convergence Theorem or Levi's Theorem: nonnegative non-decreasing and a.e.c* $\Rightarrow$ *i.c.*

4, *Lebesgue Dominated convergence Theorem:* $\exists\, g \in L(E)$, *s.t.,* $|f_n| \le g$ *and a.e.c or c.m.* $\Rightarrow$ *L.c.*

5, *Vitali Convergence Theorem: Assume that* $mE < \infty$ *and* $f_n \in L(E)$. *Then* f_n *uniformly integrable and c.m.* $\Leftrightarrow$ *L.c.*

Counterexamples:

1, *a.e.c.* or *n.u.c.* $\nrightarrow$ *u.c.* $\rhd$ $E = [0,1], f_n = x^n, f = 0$

2, *a.e.c.* $\nrightarrow$ *n.u.c.* or *c.m.* if $mE = \infty \rhd E = [0,\infty), f_n = \chi_{[0,n]}, f = 1$

3, *a.e.c.* or *n.u.c* $\nrightarrow$ *L.c.* or *i.c.*$\rhd E = [0,1], f_n = n\chi_{(0,\frac{1}{n})}, f = 0$

4, *c.m.* $\nrightarrow$ *a.e.c.*, *L.c.* $\nrightarrow$ *u.c.*, *a.e.c.* or *n.u.c*$\rhd$ $E = [0,1], f_j^{(n)} = \chi_{[\frac{j-1}{n},\frac{j}{n}]}$

5, *c.m.* $\nrightarrow$ *n.u.c.*$\rhd E = [0,\infty), f_n(x) = \chi_{[n,n+\frac{1}{n}]}, f = 0$

6, *u.c.* $\nrightarrow$ *L.c.* if $mE = \infty \rhd E = [0,\infty), f_n = \frac{1}{n}\chi_{[0,n]}, f = 0$

7, *i.c.* $\nrightarrow$ *L.c.*$\rhd$ $E = [0, 2\pi], f_n = 1 + \sin nx, f = 1$

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