EXERCISE 1

1. Suppose that f is a smooth function on $\mathbb{R}$ and that $f^{(n)} \in L^1(\mathbb{R})$ for all $n \geq 0$. Show that

$$\frac{1}{X}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{X}\right) - \int_{\mathbb{R}} f(x)dx = O(X^{-k}), \quad \forall k$$

as $X \to \infty$.

2. Let $f \in \mathcal{S}(\mathbb{R})$ and let

$$\widetilde{F}(s) = \int_0^\infty f(t)t^s \frac{dt}{t}.$$

Recall the Mellin inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}s=s_0} \widetilde{F}(s)x^{-s}ds, \qquad \text{for any } s_0 > 0.$$

a) Let $\{a_n\}_{n=1}^\infty$ be a sequence such that $a_n = O(n^k)$ for some $k > 0$. Let

$$D(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

For any $\varphi \in \mathcal{S}(\mathbb{R})$, let $S_\varphi(X)$ be the smooth sum

$$S_\varphi(X) = \sum_{n=1}^\infty a_n \varphi\left(\frac{n}{X}\right).$$

Show that

$$S_\varphi(X) = \frac{1}{2\pi i}\int_{\mathrm{Re}s=s_0} D(s)\widetilde{\varphi}(s)X^s ds$$

for $s_0 \gg 0$.

b) Suppose further that $D(s)$ can be extended to a meromorphic function with a simple pole at $s = 1$ as the only possible singularity. Suppose also that for any $a < b$, $\exists n, c$ such that $|(s-1)D(s)| \leq c(1+|s|)^n$ for all s satisfying $a \leq \mathrm{Re}s \leq b$. If $\int_{\mathbb{R}} \varphi(x)dx = 1$, then

$$\left|\sum_{n=1}^\infty a_n\varphi\left(\frac{n}{X}\right) - \mathrm{Res}_{s=1}D(s)X\right| = O(X^{-k}), \quad \forall k > 0$$

as $X \to \infty$.

c) Let $r_n = \sharp\{(k,l) \in \mathbb{Z}^2 : k^2 + l^2 = n\}$. Deduce that

$$\left| \sum_{n=1}^{\infty} r_n \varphi\left(\frac{n}{X}\right) - \pi X \right| = O(X^{-k}), \quad \forall k > 0$$

as $X \to \infty$.

3. a) From the Possion Summation Formula in the plane deduce the Hardy-Voronoi formula

$$\sum_{n=0}^{\infty} r_n h(n) = \pi \sum_{n=0}^{\infty} r_n \widetilde{h}(n), \qquad \text{for any } h \in \mathcal{S}(\mathbb{R}),$$

where

$$\widetilde{h}(y) = \int_0^{\infty} h(x) J_0(2\pi\sqrt{xy})dx,$$

and J_0 is the Bessel function

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)d\theta.$$

b) Let $R(X) = \sum_{n \le X} r_n - \pi X$ and let $\widetilde{R}(X) = \int_0^X R(t)dt$. Show that

$$\sum_{n \le X} (X-n) r_n = \frac{\pi}{2} X^2 + \widetilde{R}(X).$$

Using $h(x) = \max(0, X-x)$, show that

$$\widetilde{R}(X) = \frac{X}{\pi} \sum_{n=1}^{X} \frac{r_n}{n} J_2(2\pi\sqrt{nX}),$$

where

$$J_2(x) = \frac{2}{x^2}\int_0^x J_0(t)tdt - J_0(x).$$

c) Show that

$$O(h) + \frac{1}{h}\int_{X-h}^{X} R(t)dt \le R(X) \le O(h) + \frac{1}{h}\int_X^{X+h} R(t)dt.$$

d) Using the asymptotic behavior

$$J_2(x) = O(x^{-\frac{1}{2}}), \quad \text{as} \quad x \to \infty,$$

show that

$$\widetilde{R}(X) = O(X^{\frac{3}{4}}).$$

Deduce that

$$R(X) = O(X^{\frac{3}{8}}).$$