# Langlands picture of automorphic forms and $L$-functions

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## §2 Tate's thesis (Mar. 6)

**Theorem 2.1 (Riemann, 1859).** *The meromorphic function* $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ *defined for* $\mathrm{Re}(s)>1$ *extends analytically to all of* $\mathbb{C}$ *and satisfies*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

In 1910's, Hecke generalized Riemann's work to certain zeta functions associated to number fields. Let $F$ be a number field and $\mathcal{O}_F$ its ring of integers. We shall refer to a non-archimedean place $v$ of $F$ as a prime ideal $\mathfrak{p}\subset\mathcal{O}_F$. For a prime ideal $\mathfrak{p}$, $N\mathfrak{p}$ is the number of elements in the finite field $\mathcal{O}_F/\mathfrak{p}$. The Hecke character is defined as the product of a family of homomorphisms $\chi_v : F_v^\times \to C^\times$, i.e.,

$$\chi(x)=\prod_v \chi_v(x).$$

Note that $\chi$ is trivial on $F^\times$ and for almost all $v$, $\chi_v$ are unramified.

**Theorem 2.2 (Hecke, 1916).** *The function*

$$L(s,\chi)=\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1}$$

*defined for* $\mathrm{Re}(s)>1$ *extends analytically to all of* $\mathbb{C}$ *and satisfies*

$$A^s\Gamma\left(s,\chi\right)L(s,\chi)=W(\chi)A^{1-s}\Gamma\left(1-s,\chi^\vee\right)L(1-s,\chi^\vee),$$

*where $A$ is a constant and $W(\chi)$ is the root number.*

In 1950, Tate made use of harmonic analysis on the Adele groups to reprove both the analytic continuation and the functional equation of $L(s,\chi)$. Let's recall some notation. Obviously, $\mathbb{Q}\subset\mathbb{Q}_\infty=\mathbb{R}$ and $\mathbb{Q}\subset\mathbb{Q}_p$ for any finite $p$. We will denote

$$\mathbb{A}=\prod_{p\leqslant\infty}{}' \mathbb{Q}_p, \qquad \mathbb{A}^\times=\prod_{p\leqslant\infty}{}' \mathbb{Q}_p^\times.$$

Here the restricted product means that, for almost all $p$, $a_p\in\mathbb{Z}_p$. For any $a=(a_\infty,a_2,a_3,a_5,\ldots)\in\mathbb{A}$, $|a|_{\mathbb{A}}=\prod_{p\leqslant\infty}|a_p|_p$. The corresponding Haar measures on $\mathbb{A}$ and $\mathbb{A}^\times$ are $dx=\prod_v dx_v$ and $d^\times x=\prod_v d^\times x_v$, respectively. Moreover, we normalize matters such that

$$d^\times x_p = m_p\frac{dx_p}{|x_p|_p}, \qquad m_p=\begin{cases}1 & \text{if } p=\infty,\\ \left(1-p^{-1}\right)^{-1} & \text{if } p<\infty.\end{cases}$$

This normalization gives $\mathbb{Z}_p^\times$ volume 1 with respect to the multiplicative measure for finite $p$.

Let $c$ be a quasi-character of $\mathbb{A}^\times$, that is, a continuous homomorphism $c:\mathbb{A}^\times\to\mathbb{C}^\times$, which is trivial on $\mathbb{Q}^\times$. The quasi character $c$ can be written as $c=c_0|\cdot|_{\mathbb{A}}^s$, where $c_0:\mathbb{A}^\times\to S^1$ is a unitary character. Finally, for any $f \in \mathcal{S}(\mathbb{A})$, the Schwartz-Bruhat space, we define(see below) $f(a) = \prod_{p \leqslant \infty} f_p(a_p)$, $f_p \in \mathcal{S}(\mathbb{Q}_p)$, $a = (a_\infty, a_2, a_3, a_5, \dots) \in \mathbb{A}$.

Tate considered the zeta integrals

$$\zeta(f, c) = \int_{\mathbb{A}^\times} f(a) c(a) \, d^\times a.$$

**Theorem 2.3 (Tate, 1950).** *We have*

$$\zeta(f, c) = \zeta(\widehat{f}, c^\vee),$$

*where $\widehat{f}$ is the Fourier transform of $f$ in the adelic sense and $c^\vee(a) = \overline{c_0(a)} |a|_{\mathbb{A}}^{1-s}$.*

From the following lemma, we see that Tate's theorem recovers Riemann's theorem.

**Lemma 2.4.** *Take $c(a) = |a|_{\mathbb{A}}^s$. If we choose $f_\infty(x) = e^{-\pi x^2}$, and*

$$f_p(x) = 1_{\mathbb{Z}_p}(x) = \begin{cases} 1 & \text{if } |x|_p \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Then*

$$\zeta(f, c) = \xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

*Proof.* Folding the integral we have

$$\zeta(f, c) = \int_{\mathbb{A}^\times} f(a) c(a) \, d^\times a = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} |a|_{\mathbb{A}}^s \sum_{q \in \mathbb{Q}^\times} f(qa) \, d^\times a. \tag{2.1}$$

The *strong approximation principle* states that $(0, \infty) \times \widehat{\mathbb{Z}}^\times$ is a fundamental domain for $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, where $\widehat{\mathbb{Z}}^\times = \prod_{p < \infty} \mathbb{Z}_p^\times$. We claim that $f(qa) \equiv 0$ if the rational $q$ is not already in $\mathbb{Z}^\times$. Indeed, otherwise for any prime $p$ which divides the denominator of $q$, the $p$-adic valuation $|qa|_p = |q|_p > 1$. Here we have used the fact that $|a|_p = 1$ for $a \in \mathbb{Z}_p^\times$. Thus (2.1) becomes

$$\int_{(0,\infty) \times \widehat{\mathbb{Z}}^\times} |a|_{\mathbb{A}}^s \left( \sum_{n \in \mathbb{Z}^\times} f(na) \right) d^\times a. \tag{2.2}$$

Now $f_p((na)_p) \equiv 1$ for all $p < \infty$, and so the integrand is independent of $\widehat{\mathbb{Z}}^\times$ factor, which has volume 1 under the Haar measure. So (2.2) becomes

$$\int_0^\infty a_\infty^s \left( \sum_{n \neq 0} e^{-\pi^2 n^2 a_\infty^2} \right) d^\times a_\infty. \tag{2.3}$$

Recall that

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^s \left( \sum_{n \neq 0} e^{-\pi^2 n^2 x^2} \right) d^\times x. \tag{2.4}$$

It follows from (2.3) and (2.4) that Tate's and Riemann's integrals match for $\zeta(s)$. $\square$

**Remark 1.** Another approach to prove Lemma 2.4 is as follows. $\zeta(f, c)$ can be factored as

$$\prod_{p \leqslant \infty} \int_{\mathbb{Q}_p^\times} f_p(x) |x|_p^s \, d^\times x = \left( \int_{\mathbb{R}^\times} e^{-\pi |x|^2} |x|^s \, d^\times x \right) \cdot \prod_{p < \infty} \int_{\mathbb{Z}_p} |x_p|_p^s \, d^\times x_p. \tag{2.5}$$

We compute

$$\int_{\mathbb{R}^\times} e^{-\pi|x|^2}|x|^s\, d^\times x = \int_{\mathbb{R}} e^{-\pi|x|^2}|x|^s \frac{dx}{|x|} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

On the other hand, the $p$-adic integral may actually be broken up over a collection of "shells" $p^k\mathbb{Z}_p^\times = \{|x_p|_p = p^{-k}\}$, $k \geqslant 0$, to give the geometric series

$$\sum_{k\geqslant 0} p^{-ks} = \left(1 - p^{-s}\right)^{-1}.$$

Thus

$$\prod_{p<\infty}\int_{\mathbb{Z}_p} |x_p|_p^s\, d^\times x_p = \prod_{p<\infty}\left(1-p^{-s}\right)^{-1} = \zeta(s).$$

Combining these results with (2.5) we get

$$\zeta(f,c) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

To introduce the local theory of Tate, we need some more notation. Let $F$ be a local field with valuation $|\cdot|$. Let $dx$ be the additive Haar measure normalized so that $\int_{\mathcal{O}_F} dx = 1$. Let $d^\times x$ be the multiplicative Haar measure and $d^\times x = m\frac{dx}{|x|}$, where $m$ is a constant. We take a function $c \in \mathrm{Hom}(F^\times, \mathbb{C}^\times) = \{\text{quasi-characters of } F^\times\} = \chi(F^\times)$. Note that $\mathrm{Hom}(F^\times, S^1) = \{\text{characters of } F^\times\}$, where $S^1$ is the unit circle.

**Fact** *Every $c \in \chi(F^\times)$ can be written as $c(x) = \chi(x)|x|^s$, where $\chi$ is a unitary character.*

We will say that a complex-valued function on $F$ (or $F^\times$) is *smooth* if it is $\mathcal{C}^\infty$ for $F$ Archimedean, and locally constant otherwise. In the Archimedean case, a *Schwartz function* $f$ on $F$ is a smooth function that goes to zero rapidly at infinity. A *Schwartz-Bruhat function* is a Schwartz function of $F$ is Archimedean, and a smooth function with compact support in the non-Archimedean case. We let $\mathcal{S}(F)$ denote the space of Schwartz-Bruhat functions.

Given $f \in \mathcal{S}(F)$ and the fixed additive character $\psi$, we may define the Fourier transform of $f$ by

$$\widehat{f}(y) = \int_F f(x)\psi(xy)\, dx.$$

This maps $\mathcal{S}(F)$ onto itself. Here we assume that $\psi \in \mathrm{Hom}(F, S^1) = \widehat{F}$, i.e. every character is of the type $x \mapsto \psi_a(x) = \psi(ax)$.

**Theorem 2.5 (Tate's local theorem).** *Take*

$$\zeta(f,c) := \zeta(f,\chi,s) = \int_F f(x)c(x)\, d^\times x$$

*and let $c^\vee = c^{-1}|\cdot|$. Then*
**(A)** *This integral converges for Re$(s) = \sigma > 0$.*
**(B)** *If $\sigma \in (0,1)$, there is a functional equation*

$$\zeta(\widehat{f}, c^\vee) = \gamma(\chi,\psi,dx)\zeta(f,c)$$

*for some $\gamma(\chi,\psi,dx)$ independent of $f$, which is meromorphic as a function of $s$.*

**(C)** *For all $s \in \mathbb{C}$, there exists a nonzero factor $\varepsilon(\chi, \psi, dx)$ such that*

$$\gamma(\chi, \psi, dx) = \varepsilon(\chi, \psi, dx) \frac{L(1 - s, \chi^{-1})}{L(s, \chi)}.$$