Algebraic Number Theory Final Exam

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Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have three hours to complete this test. Good luck!*

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1, Let $k = \mathbb{Q}(\sqrt{-1}, \sqrt{5})$. Show that

(1) $\{1, \sqrt{-1}, \frac{1+\sqrt{5}}{2}, \frac{\sqrt{-1}+\sqrt{-5}}{2}\}$ is the integral base of $\mathfrak{o}_k$.

(2) There are only the rational primes 2 and 5 ramified.

(3) Find the decomposition field, inertia field, decomposition group and inertia group of (2) for $k = \mathbb{Q}(\sqrt{-1}, \sqrt{5})$ over $\mathbb{Q}$.

2, Let $k = \mathbb{Q}(\sqrt{10})$.

(1), Find the Minkowski constant $M_k = \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n}\sqrt{d_k}$.

(2), Find the class group $\mathcal{C}_k$ and the class number h_k.

3, Let $k = \mathbb{Q}(\alpha)$ with α a root of $f(x) = x^4 - 14$.

(1), Show that the prime 11 has three extensions to prime $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ of k and compute the degrees of the extensions $[k_{\mathfrak{p}_1} : \mathbb{Q}_{11}]$, $[k_{\mathfrak{p}_2} : \mathbb{Q}_{11}]$ and $[k_{\mathfrak{p}_3} : \mathbb{Q}_{11}]$.

(2), The prime 13 has four extensions to primes of k.

(3), Show the prime 5 has two extensions to primes $\mathfrak{p}_1, \mathfrak{p}_2$ of k and compute the degree of $[k_{\mathfrak{p}_1} : \mathbb{Q}_5]$ and $[k_{\mathfrak{p}_2} : \mathbb{Q}_5]$.

4, Let $k_\mathfrak{p}$ be a $\mathfrak{p}$-adic number field and π be a uniformizing parameter..

(1) The local field $k_\mathfrak{p}$ contains the $(q-1)$th roots of 1 where $q = N\mathfrak{p}$.

(2) Let μ_{q-1} be the group of roots of unity of order $q-1$. Then

$$k_\mathfrak{p}^\times = \langle \pi \rangle \times \mu_{q-1} \times (1 + \mathfrak{p}_\mathfrak{p}).$$

5. Let $\mathfrak{p}$ be a prime ideal of a number field k and $k_\mathfrak{p}$ be the local field at $\mathfrak{p}$. Let dx and $d^\times x$ be Haar measures of $k_\mathfrak{p}$ and $k_\mathfrak{p}^\times$ such that $\mathrm{vol}(\mathfrak{o}_\mathfrak{p}) = \int_{\mathfrak{o}_\mathfrak{p}} dx = 1$ and $\mathrm{vol}(U_\mathfrak{p}) = \int_{U_\mathfrak{p}} d^\times x = 1$ respectively. Let $|\cdot|_\mathfrak{p}$ be the normalized valuation of k at $\mathfrak{p}$.

(1) Show that $|\alpha|_\mathfrak{p} = \int_{\alpha\mathfrak{o}_\mathfrak{p}} dx$, i.e., $d\alpha x = |\alpha|_\mathfrak{p} dx$, for any $\alpha \in k_\mathfrak{p}^\times$.

(2) Show that $d^\times x = \frac{q}{q-1}\frac{dx}{|x|_\mathfrak{p}}$ where $q = N\mathfrak{p}$.

(3) Let $\chi_p(\alpha) = e(\lambda_p(\alpha))$ be a standard additive character on $\mathbb{Q}_p$. Compute

$$\int_{x\in\mathbb{Q}_p,\ |x|_p = p^{-k}} \chi_p(x)\mathrm{d}x.$$

6, Compute $\zeta_k(0)$ for imaginary quadratic $k = \mathbb{Q}(\sqrt{-d})$ where $d > 0$ is a square free integer. In particular, compute $\zeta(0)$ for the Reimann zeta-function.