On Some Topics in Automorphic Representations

Dihua Jiang
University of Minnesota

December, 2007

Acknowledgement

My research is supported in part by USA NSF Grants, by US-Israeli BSF Grants, and by the Chinese Academy of Sciences; and also by Project 111 at East China Normal University.

Basic Structures of Numbers

- Theorem (Foundamental Theorem of Arithmetic)

  For any $r \in \mathbb{Q}$, there is prime numbers $p_1, p_2, \cdots, p_t$ and integers $e_1, e_2, \cdots, e_t$ such that

$$r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

  This is unique up to permutation.

Basic Structures of Numbers

- Theorem (Foundamental Theorem of Arithmetic)

  For any $r \in \mathbb{Q}$, there is prime numbers $p_1, p_2, \cdots, p_t$ and integers $e_1, e_2, \cdots, e_t$ such that

$$r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

  This is unique up to permutation.

  - It is a **multiplicative structure** in terms of primes.

Basic Structures of Numbers

- Theorem (Foundamental Theorem of Arithmetic)

  For any $r \in \mathbb{Q}$, there is prime numbers $p_1, p_2, \cdots, p_t$ and integers $e_1, e_2, \cdots, e_t$ such that

$$r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

  This is unique up to permutation.

  - It is a **multiplicative structure** in terms of primes.
  - The additive structure in terms of primes should be the **Goldboch Conjecture**, which asserts the expression of even integers as sum of two primes, and is a much harder problem.

Basic Structures of Numbers

- It is much easier for kids to learn addition of numbers than the multiplication of numbers.

Basic Structures of Numbers

- It is much easier for kids to learn addition of numbers than the multiplication of numbers.
- However, it seems that the multiplication has much better structure. The local-Global principle in modern number theory is one of the good examples related to the multiplicative structure of numbers.

Basic Structures of Numbers

- It is much easier for kids to learn addition of numbers than the multiplication of numbers.
- However, it seems that the multiplication has much better structure. The local-Global principle in modern number theory is one of the good examples related to the multiplicative structure of numbers.
- From $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, to know r is equivalent to know all $p_i^{e_i}$, individually

Basic Structures of Numbers

- It is much easier for kids to learn addition of numbers than the multiplication of numbers.
- However, it seems that the multiplication has much better structure. The local-Global principle in modern number theory is one of the good examples related to the multiplicative structure of numbers.
- From $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, to know r is equivalent to know all $p_i^{e_i}$, individually
- To measure r we use the usual absolute value; and to measure $p_i^{e_i}$ we use the so called p-adic absolute value.

p-adic Absolute Value

- Given a prime p, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$.

p-adic Absolute Value

- Given a prime p, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$.
- Define the p-adic absolute value

$$|r|_p := \begin{cases} p^{-e}, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$$

p-adic Absolute Value

- Given a prime p, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$.
- Define the p-adic absolute value

$$|r|_p := \begin{cases} p^{-e}, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$$

- $|\cdot|_p$ defines a nontrivial matric on $\mathbb{Q}$.

p-adic Absolute Value

- Given a prime p, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$.
- Define the p-adic absolute value

$$|r|_p := \begin{cases} p^{-e}, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$$

- $|\cdot|_p$ defines a nontrivial matric on $\mathbb{Q}$.
- For $r \in \mathbb{Q}^\times$, we have $\prod_v |r|_v = 1$.

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.
- Take the completion, we have

$$\overline{(\mathbb{Q}, |\cdot|_\infty)} = \mathbb{R}; \quad \overline{(\mathbb{Q}, |\cdot|_p)} = \mathbb{Q}_p.$$

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.
- Take the completion, we have

$$\overline{(\mathbb{Q}, |\cdot|_\infty)} = \mathbb{R}; \quad \overline{(\mathbb{Q}, |\cdot|_p)} = \mathbb{Q}_p.$$

- They are only locally compact topological fields containing $\mathbb{Q}$ as a dense set.

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.
- Take the completion, we have

$$\overline{(\mathbb{Q}, |\cdot|_\infty)} = \mathbb{R}; \quad \overline{(\mathbb{Q}, |\cdot|_p)} = \mathbb{Q}_p.$$

- They are only locally compact topological fields containing $\mathbb{Q}$ as a dense set.
- For $v = \infty$ or p, denote the Haar measure dx_v on $\mathbb{Q}_v$, which is unique up to a constant.

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.
- Take the completion, we have

$$\overline{(\mathbb{Q}, |\cdot|_\infty)} = \mathbb{R}; \quad \overline{(\mathbb{Q}, |\cdot|_p)} = \mathbb{Q}_p.$$

- They are only locally compact topological fields containing $\mathbb{Q}$ as a dense set.
- For $v = \infty$ or p, denote the Haar measure dx_v on $\mathbb{Q}_v$, which is unique up to a constant.
- The Harmonic Analysis on $(\mathbb{Q}_v, dx_v)$ is expected to have deep impact in Number Theory.

The Riemann Zeta Function

- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re(s) > 1$.

The Riemann Zeta Function

- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re(s) > 1$.
- By the Fundamental Theorem of Arithmetic, we have the eulerian product:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

The Riemann Zeta Function

- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re(s) > 1$.
- By the Fundamental Theorem of Arithmetic, we have the eulerian product:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

- The pole at $s = 1$ of $\zeta(s)$ implies there are infinitely many primes!

The Riemann Zeta Function

- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re(s) > 1$.
- By the Fundamental Theorem of Arithmetic, we have the eulerian product:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

- The pole at $s = 1$ of $\zeta(s)$ implies there are infinitely many primes!
- The p-factor has something to do with harmonic analysis over $\mathbb{Q}_p$.

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.
- For each $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ involves finitely many primes.

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.
- For each $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ involves finitely many primes.
- The ring of adeles is defined to be

$$\mathbb{A} := \{(x_v) \in \prod_v \mathbb{Q}_v \ : \ |x_p|_p \leq 1, \text{ for almost all } p\}.$$

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.
- For each $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ involves finitely many primes.
- The ring of adeles is defined to be

$$\mathbb{A} := \{(x_v) \in \prod_v \mathbb{Q}_v \ : \ |x_p|_p \leq 1, \text{ for almost all } p\}.$$

- $\mathbb{A}$ is a locally compact ring containing all $\mathbb{Q}_v$; and $\mathbb{Q}$ is discrete in $\mathbb{A}$ such that $\mathbb{A}/\mathbb{Q}$ is compact.

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.
- For each $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ involves finitely many primes.
- The ring of adeles is defined to be

$$\mathbb{A} := \{(x_v) \in \prod_v \mathbb{Q}_v \ : \ |x_p|_p \leq 1, \text{ for almost all } p\}.$$

- $\mathbb{A}$ is a locally compact ring containing all $\mathbb{Q}_v$; and $\mathbb{Q}$ is discrete in $\mathbb{A}$ such that $\mathbb{A}/\mathbb{Q}$ is compact.
- $(\mathbb{A}, \mathbb{Q})$ is a modern analogy of the classical pair $(\mathbb{R}, \mathbb{Z})$.

Tate's Thesis

- For each v, $\exists$ a Schwartz function ϕ_v, s.t.

$$\int_{\mathbb{Q}_v^\times} \phi_v(x)|x|_v^s d^\times x_v = \begin{cases} \frac{1}{1-p^{-s}} & \text{if } v = p, \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v = \infty. \end{cases}$$

Tate's Thesis

- For each v, $\exists$ a Schwartz function ϕ_v, s.t.

$$\int_{\mathbb{Q}_v^\times} \phi_v(x)|x|_v^s d^\times x_v = \begin{cases} \frac{1}{1-p^{-s}} & \text{if } v = p, \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v = \infty. \end{cases}$$

- $\exists$ a Schwartz function $\phi = \otimes_v \phi_v$ on $\mathbb{A}$, s.t.

$$\int_{\mathbb{A}^\times} \phi(x)|x|_{\mathbb{A}}^s d^\times x = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \cdot \prod_p \frac{1}{1-p^{-s}}.$$

Tate's Thesis

- For each v, $\exists$ a Schwartz function ϕ_v, s.t.

$$\int_{\mathbb{Q}_v^\times} \phi_v(x)|x|_v^s d^\times x_v = \begin{cases} \frac{1}{1-p^{-s}} & \text{if } v = p, \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v = \infty. \end{cases}$$

- $\exists$ a Schwartz function $\phi = \otimes_v \phi_v$ on $\mathbb{A}$, s.t.

$$\int_{\mathbb{A}^\times} \phi(x)|x|_{\mathbb{A}}^s d^\times x = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \cdot \prod_p \frac{1}{1-p^{-s}}.$$

- The local-global relation in harmonic analysis approaches the local-global relation in arithmetic!

Modern Theory of Automorphic Forms

- Generalization from $GL(1)$ to general reductive algebraic groups defined over $\mathbb{Q}$.

Modern Theory of Automorphic Forms

- Generalization from $GL(1)$ to general reductive algebraic groups defined over $\mathbb{Q}$.
- Generalization from the trivial representation of $GL(1)$ to ∞-dimensional representations of adelic groups (special locally compact groups).

Modern Theory of Automorphic Forms

- Generalization from $GL(1)$ to general reductive algebraic groups defined over $\mathbb{Q}$.
- Generalization from the trivial representation of $GL(1)$ to ∞-dimensional representations of adelic groups (special locally compact groups).
- Generalization from $\zeta(s)$ to general automorphic L-functions.

Modern Theory of Automorphic Forms

- Generalization from $GL(1)$ to general reductive algebraic groups defined over $\mathbb{Q}$.
- Generalization from the trivial representation of $GL(1)$ to ∞-dimensional representations of adelic groups (special locally compact groups).
- Generalization from $\zeta(s)$ to general automorphic L-functions.
- The Langlands Programme is to figure out the deep impacts of these generalizations to Number Theory and Arithmetic.

Algebraic Groups

- Algebraic groups G are algebraic varieties with group operations which are morphisms of algebraic varieties.

Algebraic Groups

- Algebraic groups G are algebraic varieties with group operations which are morphisms of algebraic varieties.
- For simplicity, we take $G = GL_n$, SO_m, Sp_{2n}, classical groups

Algebraic Groups

- Algebraic groups G are algebraic varieties with group operations which are morphisms of algebraic varieties.
- For simplicity, we take $G = GL_n$, SO_m, Sp_{2n}, classical groups
- For example, $SO_m = \{g \in GL_m \mid {}^t g J_m g = J_m, \det g = 1\}$, with J_m defined inductively by

$$J_m := \begin{pmatrix} & & 1 \\ & J_{m-2} & \\ 1 & & \end{pmatrix}.$$

Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$.

Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$.
- The quotient $Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume.

Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$.
- The quotient $Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume.
- $L^2(G)$ denotes the space of square-integrable functions:

$$\phi \ : \ Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty.$$

Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$.
- The quotient $Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume.
- $L^2(G)$ denotes the space of square-integrable functions:

$$\phi \ : \ Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$$

  such that

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty.$$

- Such functions ϕ are (square-integrable) automorphic functions

Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$.
- The quotient $Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume.
- $L^2(G)$ denotes the space of square-integrable functions:

$$\phi \ : \ Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$$

  such that

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty.$$

- Such functions ϕ are (square-integrable) automorphic functions
- $L^2(G)$ is a $G(\mathbb{A})$-module by $g \cdot f(x) := f(xg)$.

Cuspidal Automorphic Functions

- An an automorphic functions ϕ is called **cuspidal** if

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)dn = 0$$

for almost all $g \in G(\mathbb{A})$, where N runs over the unipotent radical of all parabolic subgroups of G.

Cuspidal Automorphic Functions

- An an automorphic functions ϕ is called **cuspidal** if

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)dn = 0$$

for almost all $g \in G(\mathbb{A})$, where N runs over the unipotent radical of all parabolic subgroups of G.

- An irreducible submodule of $L^2(G)$ generated by cuspidal automorphic functions is called **cuspidal automorphic representation** of $G(\mathbb{A})$.

Cuspidal Automorphic Functions

- An an automorphic functions ϕ is called **cuspidal** if

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)dn = 0$$

  for almost all $g \in G(\mathbb{A})$, where N runs over the unipotent radical of all parabolic subgroups of G.
- An irreducible submodule of $L^2(G)$ generated by cuspidal automorphic functions is called **cuspidal automorphic representation** of $G(\mathbb{A})$.
- $L^2_c(G)$ denotes the subspace of $L^2(G)$ generated by all irredcuible cuspidal automorphic representations, which is called the **cuspidal spectrum** of $G(\mathbb{A})$.

Cuspidal Spectrum

- Theorem (Gelfand and Piatetski-Shapiro)

$$L_c^2(G) = \oplus_{\pi \in G(\mathbb{A})^\vee} m_c(\pi) V_\pi$$

with $m_c(\pi) < \infty$.

Cuspidal Spectrum

- Theorem (Gelfand and Piatetski-Shapiro)

$$L^2_c(G) = \oplus_{\pi \in G(\mathbb{A})^\vee} m_c(\pi) V_\pi$$

with $m_c(\pi) < \infty$.

  - **Problem:** For each $(\pi, V_\pi) \in G(\mathbb{A})^\vee$, determine $m_c(\pi)$.

Cuspidal Spectrum

- Theorem (Gelfand and Piatetski-Shapiro)

$$L^2_c(G) = \oplus_{\pi \in G(\mathbb{A})^\vee} m_c(\pi) V_\pi$$

with $m_c(\pi) < \infty$.

- **Problem:** For each $(\pi, V_\pi) \in G(\mathbb{A})^\vee$, determine $m_c(\pi)$.
- For classical groups, $G = SO_m$ or Sp_{2n}, the Arthur conjecture asserts that

$$m_c(\pi) \leq \begin{cases} 1, & \text{if } G = SO_{2n+1}, Sp_{2n} \\ 2, & \text{if } G = SO_{2n}. \end{cases}$$

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)
- $G = SL_2$, $m_c(\pi) \leq 1$ (Langlands-Lebasse; D. Ramkrishnan)

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)
- $G = SL_2$, $m_c(\pi) \leq 1$ (Langlands-Lebasse; D. Ramkrishnan)
- $G = SL_n (n \geq 3)$, $m_c(\pi) > 1$ (D. Blasius; E. Lapid)

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)
- $G = SL_2$, $m_c(\pi) \leq 1$ (Langlands-Lebasse; D. Ramkrishnan)
- $G = SL_n (n \geq 3)$, $m_c(\pi) > 1$ (D. Blasius; E. Lapid)
- $G = U_3$, $m_c(\pi) \leq 1$ (J. Rogawski)

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)
- $G = SL_2$, $m_c(\pi) \leq 1$ (Langlands-Lebasse; D. Ramkrishnan)
- $G = SL_n (n \geq 3)$, $m_c(\pi) > 1$ (D. Blasius; E. Lapid)
- $G = U_3$, $m_c(\pi) \leq 1$ (J. Rogawski)
- $G = G_2$, $m_c(\pi)$ unbounded (W.-T. Gan, N. Gurevich, and D.-H. Jiang; and by W.-T. Gan)

Known Cases of Cuspidal Multiplicity: $m_c(\pi)$

- $G = GL_n$, $m_c(\pi) \leq 1$ (J. Shalika; Piatetski-Shapiro)
- $G = SL_2$, $m_c(\pi) \leq 1$ (Langlands-Lebasse; D. Ramkrishnan)
- $G = SL_n (n \geq 3)$, $m_c(\pi) > 1$ (D. Blasius; E. Lapid)
- $G = U_3$, $m_c(\pi) \leq 1$ (J. Rogawski)
- $G = G_2$, $m_c(\pi)$ unbounded (W.-T. Gan, N. Gurevich, and D.-H. Jiang; and by W.-T. Gan)
- $G = GSp_4$, $m_c(\pi) \leq 1$ with π generic (D.-H. Jiang and D. Soudry)

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\longrightarrow S} \mathbb{A}(S)$,

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\overrightarrow{S}} \mathbb{A}(S)$,
- with $\mathbb{A}(S) = (\prod_{v \in S} \mathbb{Q}_v) \times (\prod_{p \notin S} \mathbb{Z}_p)$

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\substack{\longrightarrow \\ S}} \mathbb{A}(S)$,
- with $\mathbb{A}(S) = (\prod_{v \in S} \mathbb{Q}_v) \times (\prod_{p \notin S} \mathbb{Z}_p)$
- Hence $\mathbb{A}$ is a restricted direct product of $(\mathbb{Q}_v, \mathbb{Z}_v)$.

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\underset{S}{\longrightarrow}} \mathbb{A}(S)$,
- with $\mathbb{A}(S) = (\prod_{v \in S} \mathbb{Q}_v) \times (\prod_{p \notin S} \mathbb{Z}_p)$
- Hence $\mathbb{A}$ is a restricted direct product of $(\mathbb{Q}_v, \mathbb{Z}_v)$.
- Similarly, $G(\mathbb{A}) = \lim_{\underset{S}{\longrightarrow}} G(\mathbb{A}(S))$

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\overrightarrow{S}} \mathbb{A}(S)$,
- with $\mathbb{A}(S) = (\prod_{v \in S} \mathbb{Q}_v) \times (\prod_{p \notin S} \mathbb{Z}_p)$
- Hence $\mathbb{A}$ is a restricted direct product of $(\mathbb{Q}_v, \mathbb{Z}_v)$.
- Similarly, $G(\mathbb{A}) = \lim_{\overrightarrow{S}} G(\mathbb{A}(S))$
- with $G(\mathbb{A}(S)) = (\prod_{v \in S} G(\mathbb{Q}_v)) \times (\prod_{p \notin S} G(\mathbb{Z}_p))$.

Tensor Structure of Automorphic Representations

- S denotes any finite set of primes p and ∞.
- The ring of adeles $\mathbb{A} = \lim_{\overrightarrow{S}} \mathbb{A}(S)$,
- with $\mathbb{A}(S) = (\prod_{v \in S} \mathbb{Q}_v) \times (\prod_{p \notin S} \mathbb{Z}_p)$
- Hence $\mathbb{A}$ is a restricted direct product of $(\mathbb{Q}_v, \mathbb{Z}_v)$.
- Similarly, $G(\mathbb{A}) = \lim_{\overrightarrow{S}} G(\mathbb{A}(S))$
- with $G(\mathbb{A}(S)) = (\prod_{v \in S} G(\mathbb{Q}_v)) \times (\prod_{p \notin S} G(\mathbb{Z}_p))$.
- Hence $G(\mathbb{A})$ is a restricted direct product of $(G(\mathbb{Q}_v), G(\mathbb{Z}_v))$.

Tensor Structure of Automorphic Representations

- Theorem (Harish-Chandra; Bernstein)

  Each $G(\mathbb{Q}_v)$ is tame, i.e. of type I in the sense of C^*-algebras.

Tensor Structure of Automorphic Representations

- Theorem (Harish-Chandra; Bernstein)

  Each $G(\mathbb{Q}_v)$ is tame, i.e. of type I in the sense of C^*-algebras.

  - An irreducible unitary representation π of $G(\mathbb{A})$ is a restricted tensor product

$$\pi = \otimes_v \pi_v .$$

Tensor Structure of Automorphic Representations

- Theorem (Harish-Chandra; Bernstein)

  Each $G(\mathbb{Q}_v)$ is tame, i.e. of type I in the sense of C^*-algebras.

  - An irreducible unitary representation π of $G(\mathbb{A})$ is a restricted tensor product
  $$\pi = \otimes_v \pi_v.$$
  - π_v is an irreducible admissible unitary representation of $G(\mathbb{Q}_v)$ and π_v is unramified or of type I for almost all local places v of $\mathbb{Q}$.

Tensor Structure of Automorphic Representations

- Theorem (Harish-Chandra; Bernstein)

  Each $G(\mathbb{Q}_v)$ is tame, i.e. of type I in the sense of C^*-algebras.

  - An irreducible unitary representation π of $G(\mathbb{A})$ is a restricted tensor product
    $$\pi = \otimes_v \pi_v.$$
  - π_v is an irreducible admissible unitary representation of $G(\mathbb{Q}_v)$ and π_v is unramified or of type I for almost all local places v of $\mathbb{Q}$.
  - π_p is unramified if π_p has nonzero $K_p = G(\mathbb{Z}_p)$-fixed vectors.

The Satake Theory of spherical functions

- $\dim_{\mathbb{C}} V_{\pi_v}^{K_v} \leq 1$, where

$$V_{\pi_v}^{K_v} = \{u \in V_{\pi_v} \ : \ \pi_v(h)(u) = u, \textit{for all } h \in K_v\}.$$

The Satake Theory of spherical functions

- $\dim_{\mathbb{C}} V_{\pi_v}^{K_v} \leq 1$, where

$$V_{\pi_v}^{K_v} = \{u \in V_{\pi_v} \ : \ \pi_v(h)(u) = u, \textit{for all } h \in K_v\}.$$

- Irreducible unramified representations of $G(\mathbb{Q}_v)$ are parametrized by semi-simple conjugacy classes $c(\pi_v)$ in the Langlands dual group ${}^LG$, which is called the Satake parameter attached to π_v.

The Satake Theory of spherical functions

- $\dim_{\mathbb{C}} V_{\pi_v}^{K_v} \leq 1$, where

$$V_{\pi_v}^{K_v} = \{u \in V_{\pi_v} \ : \ \pi_v(h)(u) = u, \textit{for all } h \in K_v\}.$$

- Irreducible unramified representations of $G(\mathbb{Q}_v)$ are parametrized by semi-simple conjugacy classes $c(\pi_v)$ in the Langlands dual group ${}^LG$, which is called the Satake parameter attached to π_v.
- Irreducible unramified representations of $G(\mathbb{Q}_v)$ are realized as the unramified irreducible constituent of the induced representation

$$Ind_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v),$$

with unramified character χ_v of $T(\mathbb{Q}_v)$, where $B = TU$ is the Borel subgroup of G.

The Langlands Dual Group of G

- (G, B, T) determines the root datum $(X, \Delta; X^\vee, \Delta^\vee)$.

The Langlands Dual Group of G

- (G, B, T) determines the root datum $(X, \Delta; X^\vee, \Delta^\vee)$.
- Over $\mathbb{C}$, $(X, \Delta; X^\vee, \Delta^\vee)$ determines $G(\mathbb{C})$.

The Langlands Dual Group of G

- (G, B, T) determines the root datum $(X, \Delta; X^\vee, \Delta^\vee)$.
- Over $\mathbb{C}$, $(X, \Delta; X^\vee, \Delta^\vee)$ determines $G(\mathbb{C})$.
- The Langlands (complex) dual group $G^\vee(\mathbb{C})$ of G

$$\begin{array}{ccc} G & \Longleftrightarrow & (X, \Delta; X^\vee, \Delta^\vee) \\ \updownarrow & & \updownarrow \\ G^\vee(\mathbb{C}) & \Longleftrightarrow & (X^\vee, \Delta^\vee; X, \Delta) \end{array}$$

The Langlands Dual Group of G

- (G, B, T) determines the root datum $(X, \Delta; X^\vee, \Delta^\vee)$.
- Over $\mathbb{C}$, $(X, \Delta; X^\vee, \Delta^\vee)$ determines $G(\mathbb{C})$.
- The Langlands (complex) dual group $G^\vee(\mathbb{C})$ of G

$$\begin{array}{ccc} G & \Longleftrightarrow & (X, \Delta; X^\vee, \Delta^\vee) \\ \updownarrow & & \updownarrow \\ G^\vee(\mathbb{C}) & \Longleftrightarrow & (X^\vee, \Delta^\vee; X, \Delta) \end{array}$$

- $GL_n^\vee(\mathbb{C}) = GL_n(\mathbb{C})$ and $SO_{2n+1}^\vee(\mathbb{C}) = Sp_{2n}(\mathbb{C})$.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.
- For $p \notin S$, take a semisimple conjugacy calss $c_p \in G^\vee(\mathbb{C})$.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.
- For $p \notin S$, take a semisimple conjugacy calss $c_p \in G^\vee(\mathbb{C})$.
- We set $c(S) := \{c_v \mid v \notin S\}$.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.
- For $p \notin S$, take a semisimple conjugacy calss $c_p \in G^\vee(\mathbb{C})$.
- We set $c(S) := \{c_v \mid v \notin S\}$.
- For S and S', $c(S)$ and $c'(S')$ are **equivalent** if $\exists$ a set S'', containing $S \cup S'$, s.t. $c(S'') = c'(S'')$ as conjugacy classes in $G^\vee(\mathbb{C})$.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.
- For $p \notin S$, take a semisimple conjugacy calss $c_p \in G^\vee(\mathbb{C})$.
- We set $c(S) := \{c_v \mid v \notin S\}$.
- For S and S', $c(S)$ and $c'(S')$ are **equivalent** if $\exists$ a set S'', containing $S \cup S'$, s.t. $c(S'') = c'(S'')$ as conjugacy classes in $G^\vee(\mathbb{C})$.
- Denote by $\mathcal{C}(G)$ the equivalence classes of all such sets $c(S)$.

Near-Equivalence Classes

- S denotes any finite set of primes p and ∞.
- For $p \notin S$, take a semisimple conjugacy calss $c_p \in G^\vee(\mathbb{C})$.
- We set $c(S) := \{c_v \mid v \notin S\}$.
- For S and S', $c(S)$ and $c'(S')$ are **equivalent** if $\exists$ a set S'', containing $S \cup S'$, s.t. $c(S'') = c'(S'')$ as conjugacy classes in $G^\vee(\mathbb{C})$.
- Denote by $\mathcal{C}(G)$ the equivalence classes of all such sets $c(S)$.
- Denote by $\mathcal{A}(G)$ the set of irreducible cuspidal automorphic representations of $G(\mathbb{A})$ up to equivalence.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c : \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c \ : \ \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.
- $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are of **near-eqivalence** if for almost all primes p, π_p and π'_p are equivalent.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c \ : \ \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.
- $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are of **near-eqivalence** if for almost all primes p, π_p and π'_p are equivalent.
- **Problems:**

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c \ : \ \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.
- $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are of **near-eqivalence** if for almost all primes p, π_p and π'_p are equivalent.
- **Problems:**
- (1) Describe the image $c(\mathcal{A}(G))$ in $\mathcal{C}(G)$.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c : \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.
- $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are of **near-eqivalence** if for almost all primes p, π_p and π'_p are equivalent.
- **Problems:**
- (1) Describe the image $c(\mathcal{A}(G))$ in $\mathcal{C}(G)$.
- (2) Describe the fibre $\Pi_{c(\pi)}$.

Near-Equivalence Classes

- For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, $\exists$ an S_π s.t. for $p \notin S_\pi$, π_p is unramified. Define $c(\pi) := c(S_\pi)$.
- $\exists$ a map $c \ : \ \pi \mapsto c(\pi)$ from $\mathcal{A}(G)$ to $\mathcal{C}(G)$. The fibre $\Pi_{c(\pi)}$ is called the **nearly equivalence classes** of π.
- $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are of **near-eqivalence** if for almost all primes p, π_p and π'_p are equivalent.
- **Problems:**
- (1) Describe the image $c(\mathcal{A}(G))$ in $\mathcal{C}(G)$.
- (2) Describe the fibre $\Pi_{c(\pi)}$.
- (3) Determine the structures of π in terms of $c(\pi)$.

Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)
  For $G = GL_n$, $\Pi_{c(\pi)}$ contains one member. (For π, π' in $\mathcal{A}(G)$, if $c(\pi) = c(\pi')$, then π, π' are equivalent.)

Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)

  For $G = GL_n$, $\Pi_{c(\pi)}$ contains one member. (For π, π' in $\mathcal{A}(G)$, if $c(\pi) = c(\pi')$, then π, π' are equivalent.)

- Theorem (Jiang-Soudry, 2003)

  For $G = SO_{2n+1}$, $\Pi_{c(\pi)}$ contains at most one generic member; and if π is tempered, $\Pi_{c(\pi)}$ contains at least one generic member.

Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)

  For $G = GL_n$, $\Pi_{c(\pi)}$ contains one member. (For π, π' in $\mathcal{A}(G)$, if $c(\pi) = c(\pi')$, then π, π' are equivalent.)

- Theorem (Jiang-Soudry, 2003)

  For $G = SO_{2n+1}$, $\Pi_{c(\pi)}$ contains at most one generic member; and if π is tempered, $\Pi_{c(\pi)}$ contains at least one generic member.

  - For $G = SO_{2n+1}$, if two generic π, π' in $\mathcal{A}(G)$ are of near-equivalence, then π, π' are equivalent. (rigidity)

Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)

  For $G = GL_n$, $\Pi_{c(\pi)}$ contains one member. (For π, π' in $\mathcal{A}(G)$, if $c(\pi) = c(\pi')$, then π, π' are equivalent.)

- Theorem (Jiang-Soudry, 2003)

  For $G = SO_{2n+1}$, $\Pi_{c(\pi)}$ contains at most one generic member; and if π is tempered, $\Pi_{c(\pi)}$ contains at least one generic member.

  - For $G = SO_{2n+1}$, if two generic π, π' in $\mathcal{A}(G)$ are of near-equivalence, then π, π' are equivalent. (rigidity)
  - It is important to the Arthur trace formula approach.

Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)

  For $G = GL_n$, $\Pi_{c(\pi)}$ contains one member. (For π, π' in $\mathcal{A}(G)$, if $c(\pi) = c(\pi')$, then π, π' are equivalent.)

- Theorem (Jiang-Soudry, 2003)

  For $G = SO_{2n+1}$, $\Pi_{c(\pi)}$ contains at most one generic member; and if π is tempered, $\Pi_{c(\pi)}$ contains at least one generic member.

  - For $G = SO_{2n+1}$, if two generic π, π' in $\mathcal{A}(G)$ are of near-equivalence, then π, π' are equivalent. (rigidity)
  - It is important to the Arthur trace formula approach.
  - A slight modification holds for general reductive groups. For classical groups, it is my on-going joint work with D. Soudry.

Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both π_p and τ_p are unramified.

Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both π_p and τ_p are unramified.
- Define the (partial) Rankin-Selberg convolution L-function by

$$L^S(s, \pi \times \tau) := \prod_{p \notin S} \frac{1}{\det(I - c(\pi_p) \otimes c(\tau_p) p^{-s})}.$$

Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both π_p and τ_p are unramified.
- Define the (partial) Rankin-Selberg convolution L-function by

$$L^S(s, \pi \times \tau) := \prod_{p \notin S} \frac{1}{\det(I - c(\pi_p) \otimes c(\tau_p) p^{-s})}.$$

- When G is classical, $L^S(s, \pi \times \tau)$ has meromorphic continuation and functional equation.

Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both π_p and τ_p are unramified.
- Define the (partial) Rankin-Selberg convolution L-function by

$$L^S(s, \pi \times \tau) := \prod_{p \notin S} \frac{1}{\det(I - c(\pi_p) \otimes c(\tau_p) p^{-s})}.$$

- When G is classical, $L^S(s, \pi \times \tau)$ has meromorphic continuation and functional equation.
- **Problem:** Determine the poles of $L^S(s, \pi \times \tau)$ for $s \geq \frac{1}{2}$.

Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both π_p and τ_p are unramified.
- Define the (partial) Rankin-Selberg convolution L-function by

$$L^S(s, \pi \times \tau) := \prod_{p \notin S} \frac{1}{\det(I - c(\pi_p) \otimes c(\tau_p) p^{-s})}.$$

- When G is classical, $L^S(s, \pi \times \tau)$ has meromorphic continuation and functional equation.
- **Problem:** Determine the poles of $L^S(s, \pi \times \tau)$ for $s \geq \frac{1}{2}$.
- This is closely related to the structures of $c(\pi)$ and π, i.e. the local-global relations.

Langlands Functorial Transfers

- **Weak Langlands Transfer Conjecture:** *Let G and H be k-split reductive algebraic groups and let ρ be any group homomorphism*

$$\rho \ : \ H^\vee(\mathbb{C}) \to G^\vee(\mathbb{C}).$$

For any $\sigma \in \mathcal{A}(H)$, $\exists$ a $\pi \in \mathcal{A}(G)$ (may not be cuspidal!) s.t.

$$c(\rho(\sigma)) = c(\pi)$$

as conjugacy classes in $G^\vee(\mathbb{C})$, where $c(\rho(\sigma)) = \{\rho(c(\sigma_v))\}$.

Langlands Functorial Transfers

- **Weak Langlands Transfer Conjecture:** *Let G and H be k-split reductive algebraic groups and let ρ be any group homomorphism*

$$\rho \;:\; H^\vee(\mathbb{C}) \to G^\vee(\mathbb{C}).$$

For any $\sigma \in \mathcal{A}(H)$, $\exists$ a $\pi \in \mathcal{A}(G)$ (may not be cuspidal!) s.t.

$$c(\rho(\sigma)) = c(\pi)$$

as conjugacy classes in $G^\vee(\mathbb{C})$, where $c(\rho(\sigma)) = \{\rho(c(\sigma_v))\}$.

- The **strong Langlands Functorial Transfer** requires compatibility at all local palces or can be stated in terms of the complete tensor product L-functions.

Existence of the Weak Langlands Transfers

- Arthur-Clozel (1989) and Badulescu (2007): Generalized Jacquct-Langlands transfer between GL_n and its inner forms.

Existence of the Weak Langlands Transfers

- Arthur-Clozel (1989) and Badulescu (2007): Generalized Jacquct-Langlands transfer between GL_n and its inner forms.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2001, 2004): Langlands transfer from classical groups and GL_n-type.

Existence of the Weak Langlands Transfers

- Arthur-Clozel (1989) and Badulescu (2007): Generalized Jacqcut-Langlands transfer between GL_n and its inner forms.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2001, 2004): Langlands transfer from classical groups and GL_n-type.
- Kim-Krishnamurthy (2004, 2005): $U(n, n)$ and $U(n+1, n)$.

Existence of the Weak Langlands Transfers

- Arthur-Clozel (1989) and Badulescu (2007): Generalized Jacquct-Langlands transfer between GL_n and its inner forms.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2001, 2004): Langlands transfer from classical groups and GL_n-type.
- Kim-Krishnamurthy (2004, 2005): $U(n, n)$ and $U(n+1, n)$.
- Asgari-Shahidi (2006): $GSpin_m$.

Existence of the Weak Langlands Transfers

- Arthur-Clozel (1989) and Badulescu (2007): Generalized Jacqcut-Langlands transfer between GL_n and its inner forms.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2001, 2004): Langlands transfer from classical groups and GL_n-type.
- Kim-Krishnamurthy (2004, 2005): $U(n, n)$ and $U(n+1, n)$.
- Asgari-Shahidi (2006): $GSpin_m$.
- Gelbart-Jacquet (1978): $Sym^2(GL_2)$; Kim-Shahidi (2002): $Sym^3(GL_2)$: Kim (2003): $Sym^4(GL_2)$; Ramakrishnan (2000): $GL_2 \otimes GL_2$; Kim-Shahidi (2002): $GL_2 \otimes GL_3$; Ginzburg-Jiang (2001): $G_2 \to GSp_6$; Ginzburg (2005): $GL_2 \times GL_2 \to G_2$.

Refined Properties of Langlands Transfers

- **Local-Global Compatibility:**

Refined Properties of Langlands Transfers

- **Local-Global Compatibility:**
- Jiang-Soudry (2003): $SO_{2n+1} \Longrightarrow GL_{2n}$; With explicit local descent, we obtain the local Langlands reciprocity map for SO_{2n+1}.

Refined Properties of Langlands Transfers

- **Local-Global Compatibility:**
- Jiang-Soudry (2003): $SO_{2n+1} \Longrightarrow GL_{2n}$; With explicit local descent, we obtain the local Langlands reciprocity map for SO_{2n+1}.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2004): SO_{2n} and Sp_{2n}; The local descent in these cases are the work in progress of Jiang-Soudry, which also implies the existence of the local Langlands reciprocity map.

Refined Properties of Langlands Transfers

- **Local-Global Compatibility:**
- Jiang-Soudry (2003): $SO_{2n+1} \Longrightarrow GL_{2n}$; With explicit local descent, we obtain the local Langlands reciprocity map for SO_{2n+1}.
- Cogdell, Kim, Piatetski-Shapiro, and Shahidi (2004): SO_{2n} and Sp_{2n}; The local descent in these cases are the work in progress of Jiang-Soudry, which also implies the existence of the local Langlands reciprocity map.
- Some other cases are known, but I omit the details here.

Refined Properties of Langlands Transfers

- **Image of the Langlands Transfers:**

Refined Properties of Langlands Transfers

- **Image of the Langlands Transfers:**
- Ginzburg-Rallis-Soudry automorphic descent from GL to classical groups characterizes the image of the Langlands transfer from classical groups to GL (a series of papers in 1997-2005)
- Jiang-Soudry (2003) prove the irreducibility of the image of the descent for SO_{2n+1}; the other cases are our work in progress.

Refined Properties of Langlands Transfers

- **Image of the Langlands Transfers:**
- Ginzburg-Rallis-Soudry automorphic descent from *GL* to classical groups characterizes the image of the Langlands transfer from classical groups to *GL* (a series of papers in 1997-2005)
- Jiang-Soudry (2003) prove the irreducibility of the image of the descent for SO_{2n+1}; the other cases are our work in progress.
- C. Khare, M. Larsen, and G. Savin (2007): Use our result to study the Inverse Galois Problem over $\mathbb{Q}$.

Endoscopy and Poles of Certain L-functions

- Theorem (Jiang 2006)

  Let $\pi \in \mathcal{A}(SO_{2n+1})$ be cuspidal and generic.

  1. *The 2nd fundamental L-function $L(s, \pi, \omega_2)$ is holomorphic for $Re(s) \geq \frac{1}{2}$ with possible pole at $s = 1$*
  2. *The order of the pole at s=1 of $L(s, \pi, \omega_2)$ is $r - 1$ if and only if $\exists$ a partition $n = \sum_{i=1}^{r} n_i$ s.t. π is an endoscopy transfer from the elliptic endoscopy group*

$$SO_{2n_1+1} \times \cdots \times SO_{2n_r+1}.$$

Endoscopy and Poles of Certain L-functions

- Theorem (Jiang 2006)

  Let $\pi \in \mathcal{A}(SO_{2n+1})$ be cuspidal and generic.

  1. *The 2nd fundamental L-function $L(s, \pi, \omega_2)$ is holomorphic for $Re(s) \geq \frac{1}{2}$ with possible pole at $s = 1$*
  2. *The order of the pole at s=1 of $L(s, \pi, \omega_2)$ is $r - 1$ if and only if $\exists$ a partition $n = \sum_{i=1}^{r} n_i$ s.t. π is an endoscopy transfer from the elliptic endoscopy group*

  $$SO_{2n_1+1} \times \cdots \times SO_{2n_r+1}.$$

  - It is the work in progress of Ginzburg-Jiang to characterize the endoscopy transfers in terms of period of π, which will generalize our preliminary work in this aspect in 2001.

The Generalized Ramanujan Conjecture

- **GRC:** Any irreducible cuspidal automorphic representation is tempered

The Generalized Ramanujan Conjecture

- **GRC:** Any irreducible cuspidal automorphic representation is tempered
- R. Howe and Piatetski-Shapiro (1977): **GRC** is not true for $G \neq GL$.

The Generalized Ramanujan Conjecture

- **GRC:** Any irreducible cuspidal automorphic representation is tempered
- R. Howe and Piatetski-Shapiro (1977): **GRC** is not true for $G \neq GL$.
- One of the refinements (Jiang, 2007): *Any irreducible cuspidal automorphic representation with one local generic component is tempered.*

The Generalized Ramanujan Conjecture

- **GRC:** Any irreducible cuspidal automorphic representation is tempered
- R. Howe and Piatetski-Shapiro (1977): **GRC** is not true for $G \neq GL$.
- One of the refinements (Jiang, 2007): *Any irreducible cuspidal automorphic representation with one local generic component is tempered*.
- This formulation holds for all known examples and is compatible with the Arthur conjecture on the discrete automorphic spectrum in general.

The CAP Conjecture

Assume that G is $\mathbb{Q}$-quasisplit reductive group and G' be a $\mathbb{Q}$-inner form of G. For any irreducible cuspidal automorphic representation π' of $G'(\mathbb{A})$, there exist a standard parabolic subgroup $P = MN$ of G, an irreducible generic unitary cuspidal automorphic representation σ of $M(\mathbb{A})$, and an unramified character χ of $M(\mathbb{A})^1\backslash M(\mathbb{A})$, such that π' is nearly equivalent to an irreducible constituent of the unitarily induced representation

$$Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes \chi).$$

The CAP Conjecture

- If P is proper parabolic in G, π' is called a CAP representation of G'.

The CAP Conjecture

- If P is proper parabolic in G, π' is called a CAP representation of G'.
- The CAP representations are counter-examples to **GRC**, but is essential to understand the Arthur conjecture on the discrete automorphic spectrum.

The CAP Conjecture

- If P is proper parabolic in G, π' is called a CAP representation of G'.
- The CAP representations are counter-examples to **GRC**, but is essential to understand the Arthur conjecture on the discrete automorphic spectrum.
- Jiang-Soudry (2007): For $G = SO_{2n+1}$, the CAP datum (M, σ, χ) is determined by π', which is generalization of the rigidity of cuspidal automorphic representations.

The CAP Conjecture

- If P is proper parabolic in G, π' is called a CAP representation of G'.
- The CAP representations are counter-examples to **GRC**, but is essential to understand the Arthur conjecture on the discrete automorphic spectrum.
- Jiang-Soudry (2007): For $G = SO_{2n+1}$, the CAP datum (M, σ, χ) is determined by π', which is generalization of the rigidity of cuspidal automorphic representations.
- For other classical groups, suitable modifications are needed, which is the work in progress of Jiang-Soudry.

The CAP Conjecture

- Jacquet-Shalika (1981): the CAP conjecture holds for GL_n.

The CAP Conjecture

- Jacquet-Shalika (1981): the CAP conjecture holds for GL_n.
- A. Badulescu (2007): it holds for $GL_m(D)$, where D is a division algebra.

The CAP Conjecture

- Jacquet-Shalika (1981): the CAP conjecture holds for GL_n.
- A. Badulescu (2007): it holds for $GL_m(D)$, where D is a division algebra.
- Jiang-Soudry (2007): it holds for cuspidal automorphic representations of SO_{2n+1} with special Bessel models.

The CAP Conjecture

- Jacquet-Shalika (1981): the CAP conjecture holds for GL_n.
- A. Badulescu (2007): it holds for $GL_m(D)$, where D is a division algebra.
- Jiang-Soudry (2007): it holds for cuspidal automorphic representations of SO_{2n+1} with special Bessel models.
- Gelbart-Rogawski-Soudry (1997): it holds for $U(3)$.

The CAP Conjecture

- Jacquet-Shalika (1981): the CAP conjecture holds for GL_n.
- A. Badulescu (2007): it holds for $GL_m(D)$, where D is a division algebra.
- Jiang-Soudry (2007): it holds for cuspidal automorphic representations of SO_{2n+1} with special Bessel models.
- Gelbart-Rogawski-Soudry (1997): it holds for $U(3)$.
- Many families of CAP representations have been constructed, but we omit the details here.

Final Remarks

- The modern theory of automorphic forms is to understand the spectrum of $L^2(G)$ as representation of $G(\mathbb{A})$.

Final Remarks

- The modern theory of automorphic forms is to understand the spectrum of $L^2(G)$ as representation of $G(\mathbb{A})$.
- The L-function and the Converse Theorem approach gives the information about $L^2(G)$ via specific families of spectrum, but by constructive methods, based on L-functions.

Final Remarks

- The modern theory of automorphic forms is to understand the spectrum of $L^2(G)$ as representation of $G(\mathbb{A})$.
- The L-function and the Converse Theorem approach gives the information about $L^2(G)$ via specific families of spectrum, but by constructive methods, based on L-functions.
- The Arthur-Selberg trace formula gets the complete structure of the spectrum, which yields the existence of endoscopy transfers in general, and has many potential applications.

Final Remarks

- The modern theory of automorphic forms is to understand the spectrum of $L^2(G)$ as representation of $G(\mathbb{A})$.
- The L-function and the Converse Theorem approach gives the information about $L^2(G)$ via specific families of spectrum, but by constructive methods, based on L-functions.
- The Arthur-Selberg trace formula gets the complete structure of the spectrum, which yields the existence of endoscopy transfers in general, and has many potential applications.
- The rational combination of the Arthur trace formula with the L-function and the Converse Theorem methods is definitely a very interesting approach for the near future.