Lie Groups and Representations for Undergraduate

Final Exam, Spring 2013

School of Mathematics, Shandong University

Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two hours to complete this test. Good luck!*

Let E be a n-dimensional real or complex vector space and $M = L(E)$ be the space of all linear transformations of E, or $n \times n$ matrix space.

—— 14:00-16:30, June 17, 2013.

***1.*(10 points)** Suppose $X \in M$ satisfies $\|X\| < 2\pi$. Show that $v \in E$,

$$\exp(X)v = v \quad \text{if and only if} \quad Xv = 0.$$

***2.*(20 points)** (1), Show that for any $X, Y \in M$, small $t \in \mathbb{R}$,

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right),$$

(2), Show that

$$\lim_{k\to\infty}\left(\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right)^k = \exp(X+Y).$$

***3.*(20 points)** Let G and H be linear Lie groups with linear Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose that $\Phi: G \longrightarrow H$ is a differentiable homomorphism of linear Lie groups. Then there exists a unique real linear map $\phi: \mathfrak{g} \longrightarrow \mathfrak{h}$ such that, for all $X \in \mathfrak{g}$,

$$\Phi(\exp(X)) = \exp(\phi(X)).$$

The map ϕ has the following properties: for any $X, Y \in \mathfrak{g}, a \in G, t \in \mathbb{R}$,

(1), $\phi(gXg^{-1}) = \Phi(g)\phi(X)\Phi(g)^{-1}$;

(2), $\phi([X,Y]) = [\phi(X), \phi(Y)]$;

(3), $\phi(X) = \frac{d}{dt}\Phi(\exp(tX))|_{t=0}$.

***4.*(15 points)** Let G be a linear Lie group and its Lie algebra

$$\mathfrak{g} = \{X \in M(n,\mathbb{C}) \mid \exp(tX) \in G \text{ for any } t \in \mathbb{R}\}.$$

(1), Find the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ and the dimension of the group $SL(n,\mathbb{C})$;

(2), Find the Killing form of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$;

(3), Show that $\mathfrak{sl}(3,\mathbb{C})$ is a simple Lie algebra.

***5.*(15 points)** Let G be a compact matrix group and π be a finite dimensional complex representation of G. Show that $|\chi_\pi(g)| \leq \chi_\pi(1)$ with equality only of $\pi(g)$ is a scalar.

***6.*(20 points)** (1), Show that the elements of $SU(2)$ are of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \ |\alpha|^2 + |\beta|^2 = 1.$$

(2), Let

$$V_m = \left\{ f(z_1, z_2) = \sum_{i=0}^{m} a_i z_1^{m-i} z_2^i \mid a_i \in \mathbb{C} \right\}$$

be the space of homogeneous polynomials in two complex variables with total degree $m (m \geq 0)$. Denote a linear transformation $\pi(g)$ on the space V_m by the formula

$$(\pi(g)(f))(z) = f(g^{-1}z),$$

for any $g \in SU(2)$ and any $z = (u_1, u_2)^T \in \mathbb{C}^2$. Show that π is an $m+1$ dimensional complex representation of the group $SU(2)$ on the vector space V_m.

(3), Let χ_π be the character of the representation π and $t_\theta = \text{diag}\{e^{i\theta}, e^{-i\theta}\} \in SU(2)$. Compute $\chi_\pi(t_\theta)$.

(4), Let $W \subset V_m$ be a nonzero invariant subspace. Show that if $z_1^{m-k} z_2^k \in W$ for any fixed $0 \leq k \leq m$, then $z_1^{m-k-1} z_2^{k+1}, z_1^{m-k+1} z_2^{k-1} \in W$. And then π is irreducible.