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2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, y in X. Show that f is an onto map.

5. Let $K(x, y) \in C([0, 1] \times [0, 1])$. For all $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y)f(y)dy$$

Prove that $Tf \in C([0, 1])$. Moreover $\Omega = \{Tf| \, ||f||_{sup} \leq 1\}$ is precompact in $C([0, 1])$, i.e. every sequence in Ω has a converging subsequence, here $||f||_{sup} = \sup\{|f(x)||x \in [0, 1]\}$.

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3,

Let $T : C_{\mathbb{R}}[0, 1] \to \mathbb{R}$ be the bounded linear transformation defined by

$$T(f) = \int_0^1 f(x)dx.$$

(a) Show that $\|T\| \leq 1$.

(b) If $g \in C_{\mathbb{R}}[0, 1]$ is defined by $g(x) = 1$ for all $x \in [0, 1]$, find $|T(g)|$ and hence find $\|T\|$.

4,

Let $T : \ell^2 \to \ell^2$ be the bounded linear transformation defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (0, 4x_1, x_2, 4x_3, x_4, \ldots).$$

Find the norm of T. and

Find T^2.

Hence find $\|T^2\|$ and compare this with $\|T\|^2$.

5,

设 $(Tu)(x) = xu(x)$, $(Su)(x) = x\int_0^1 u(y)\mathrm{d}y$, $u \in C[0,1]$, 求 $\|T\|$, $\|S\|$, $\|TS\|$, $\|ST\|$.

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6,

设 $T_n x = (x_1, \cdots, x_n, 0, \cdots)$, 则 $T \in L(l^2)$, 求 $\|T_n\|$.

记注: 求范数, 详见胡适耕教材

2. 算子范数 对于 $T \in L(X,Y)$, 范数 $\|T\|$ 无非是实泛函 $\|Tx\|$ 在闭单位球上的上确界, 在某些特殊情况下(例如 $\dim X < \infty$, 但不限于这种情况), 该上确界实际上是极大值. 无论迫于实际的需要, 或者基于追求理论上完整性的偏好, 人们常常致力于准确地求出所研究的线性算子的范数, 但这未必容易成功. 读者在面对这类问题时, 通常有两种方法可供选择:

(A) 利用已知的标准结果, 例如本章的 2.1.3, 2.2.1 ~ 2.2.4, 以及下章的 3.5.7(i), 3.6.2 等. 这类结果在泛函分析中为数不多, 其作用是有限的.

(B) 直接法. 通常依如下程序进行: $\forall x \in X$, 对 $\|Tx\|$ 作出一个尽可能准确的估计 $\|Tx\| \leqslant k\|x\|$, 从而推测 $\|T\| = k$. 为证实这一推测, 可用以下方法之一:

(i) 选取适当的 $x_0 \in X$, 使 $\|x_0\| = 1$, $\|Tx_0\| = k$, 如在 2.1.3 与 2.7.7 中即是如此. 这一方法显然仅当 $\|T\| = \max\limits_{\|x\|=1}\|Tx\|$ 时才有效, 因此应用极有限.

(ii) $\forall \alpha < k$, 选取 $x_\alpha \in X$, 使 $\|x_\alpha\| = 1$, $\|Tx_\alpha\| \geqslant \alpha$, 如在命题 2.2.2 之证中所作的.

(iii) 选取 $x_n \in X$, 使 $\|x_n\| = 1$, $\varlimsup\limits_n \|Tx_n\| \geqslant k$. 2.2.4 之证即用此法.

毫无疑问, 以上方法成功的前提是 $\|T\| = k$ 是一正确猜测, 而这是无法保证的. 如本章曾指出的, 不能准确地求出 $\|T\|$, 未必总是一个很严重的问题. 例如, 对于 2.2.2(iii) 中所述的算子 $A \in L(l^p)$, 尽管未曾求得 $\|A\|_p$ 的准确表达

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7,

设X, Y为Banach空间, $T_n \in L(X, Y)$. 证明以下命题互相等价:
(i), $\{\|T_n\|\}$ 有界;
(ii), $\{\|T_n x\|\}$ 对任$x \in X$有界;
(iii), $\{|f(T_n x)|\}$对任$x \in X, f \in X'$有界.

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8,

Suppose that X is a Banach space, Y is a normed space and $T \in B(X, Y)$. If there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in X$, then $\mathrm{Im}(T)$ is closed.

2011/11/30,

9,

6.16 Let $T \in B(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_1, -x_2, x_3, -x_4, \ldots).$$

(a) Show that 1 and -1 are eigenvalues of T with eigenvectors $(1, 0, 0, \ldots)$ and $(0, 1, 0, 0, \ldots)$ respectively.

(b) Find T^2 and hence show that $\sigma(T) = \{-1, 1\}$.

10,

6. 设 $T: l^1 \to l^1, T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$, 证明 $\sigma_p(T) = \emptyset$, $\sigma(T) = \{\lambda : |\lambda| \leqslant 1\}$.

2011/12/05,

11,

设 $Tf(x) = \int_a^x f(t)\mathrm{d}t, f \in C[a,b]$, 求 $\sigma(T)$.

12,

设 $Tf(x) = xf(x), f \in C[0,1]$, 求 $\sigma(T)$ 与 $\sigma_p(T)$.

注记: 求谱集, 详见胡适耕"实变函数与泛函分析定理方法问题"

现在初步总结一下. 为求 $T \in L(X)$ 的谱 $\sigma(T)$, 可循以下步骤:

(i) 若容易确定 $r_\sigma(T) = 0$, 则 $\sigma(T) = \{0\}$, 问题解毕.

(ii) 任取 $\lambda \in \mathbf{C}, y \in X$, 研究方程

$$\lambda x - Tx = y$$

的可解性. 若以上方程有唯一解 x_λ, 且 x_λ 可表为 $x_\lambda = A_\lambda y, A_\lambda \in L(X)$, 则 $\lambda \in \rho(T)$, 因而 $\lambda \overline{\in} \sigma(T)$.

(iii) 若对某个 $y \in X$ 方程 $\lambda x - Tx = y$ 无解, 则必 $x \in \sigma(T)$.

(iv) 若方程 $\lambda x = Tx$ 有非零解 x, 则 $\lambda \in \sigma_p(T)$.

根据具体情况, 也可以首先考虑方程 $\lambda x = Tx$, 然后再考虑对应的非齐次方程.

2011/12/12

13,

7.11 Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and let $\{e_n\}$, $\{f_n\}$, be orthonormal sequences in $\mathcal{H}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{C}$ and define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \sum_{n=1}^{\infty} \alpha_n (x, e_n) f_n.$$

Show that:

(a) T is bounded if and only if the sequence $\{\alpha_n\}$ is bounded;

(b) T is compact if and only if $\lim_{n\to\infty} \alpha_n = 0$;

(c) T is Hilbert–Schmidt if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$;

(d) T has finite rank if and only if there exists $N \in \mathbb{N}$ such that $\alpha_n = 0$ for $n \geq N$.

It follows that each of these classes of operators is strictly contained in the preceding class. In particular, not all compact operators are Hilbert–Schmidt.