History of irrational and transcendental numbers

Michel Waldschmidt

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Abstract

The transcendence proofs for constants of analysis are essentially all based on the seminal work by Ch. Hermite : his proof of the transcendence of the number e in 1873 is the prototype of the methods which have been subsequently developed. We first show how the founding paper by Hermite was influenced by earlier authors (Lambert, Euler, Fourier, Liouville), next we explain how his arguments have been expanded in several directions : Padé approximants, interpolation series, auxiliary functions.

Numbers : rational, irrational

Numbers = real or complex numbers $\mathbf{R}$, $\mathbf{C}$.

Natural integers : $\mathbf{N} = \{0, 1, 2, \ldots\}$.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Rational numbers :
a/b with a and b rational integers, $b > 0$.

Irreducible representation :
p/q with p and q in $\mathbf{Z}$, $q > 0$ and $\gcd(p, q) = 1$.

Irrational number : a real (or complex numbers) which is not rational.

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Numbers : algebraic, transcendental

Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

Examples :
- rational numbers : a/b is root of $bX - a$.
- $\sqrt{2}$, root of $X^2 - 2$.
- i, root of $X^2 + 1$.

The sum and the product of algebraic numbers are algebraic numbers. The set $\overline{\mathbf{Q}}$ of complex algebraic numbers is a field, the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

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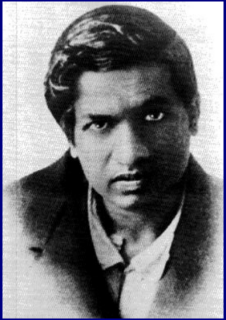
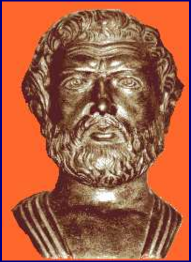
Srinivasa Ramanujan

Some transcendental aspects
of Ramanujan's work.
Proceedings of the Ramanujan Centennial International Conference
(Annamalainagar, 1987),
RMS Publ., **1**, Ramanujan Math.
Soc., Annamalainagar, 1988, 67–76.



Irrationality of $\sqrt{2}$

Pythagoreas school



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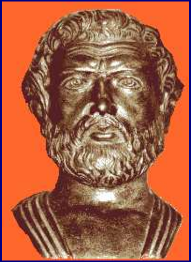

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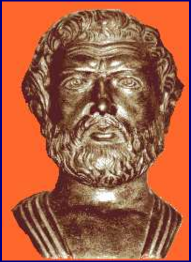

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Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion

$$\frac{1}{\sqrt{2}-1}=1+\sqrt{2},$$

which is the same as for the large one.
- Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
- This process does not end.

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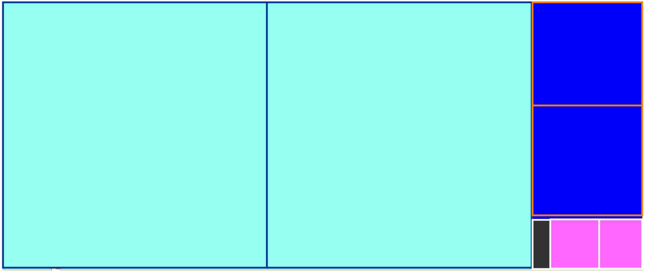
Rectangles with proportion $1+\sqrt{2}$



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If we start with a rectangle having rational side lengths, then this process stops after finitely may steps (the sequence of denominators of the side lengths does not increase, while the sequence of numerators decrease).

Hence $1+\sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

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The fabulous destiny of $\sqrt{2}$

- Benoît Rittaud, Éditions *Le Pommier* (2006).

http://www.math.univ-paris13.fr/~rittaud/RacineDeDeux

Continued fraction

The number

$$\sqrt{2} = 1,414\,213\,562\,373\,095\,048\,801\,688\,724\,209 \quad \dots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \dots] = [1;\ \overline{2}].$$

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- H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

Irrationality criteria

A real number is rational if and only if its continued fraction expansion is finite.

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To construct irrational (even transcendental) numbers is easy. To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge.

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Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right)$$
$$= 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is it a rational number ?

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right) = \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx$$
$$= -\int_0^1 \int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Recent work by *J. Sondow* inspired by the work of F. Beukers on Apéry's proof.

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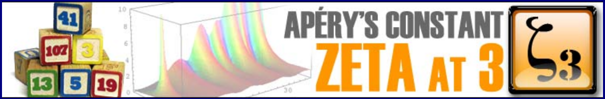

The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

is irrational (Apéry 1978).

Is-it the same for

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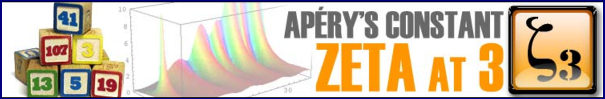

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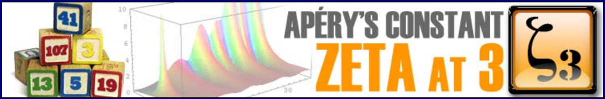

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Open problems (irrationality)

- Is the number

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- Is the number

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Euler Gamma function

Is the number

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$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

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Irrationality of π

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Continued fraction expansion of $\tan(x)$

$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot$$

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e is neither rational (J-H. Lambert, 1766) nor quadratic irrational (J-L. Lagrange, 1770).

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Continued fraction expansion for $e^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.

This leads Euler to

$$e^{1/a} = [1\,;\, a-1,\ 1,\ 1,\ 3a-1,\ 1,\ 1,\ 5a-1, \ldots]$$
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Geometric proof of the irrationality of e

Jonathan Sondow
http://home.earthlink.net/∼jsondow/
A geometric proof that e is irrational and a new measure of its irrationality,
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Start with an interval I_1 with length 1. The interval I_n will be obtained by splitting the interval I_{n-1} into n intervals of the same length, so that the length of I_n will be $1/n!$.

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The origin of I_n will be

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}.$$

Hence we start from the interval $I_1=[2,3]$. For $n\geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

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Irrationality of e, following J. Sondow

The origin of I_n is

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The number e is the intersection point of all these intervals, hence it is inside each I_n, therefore it cannot be written $a/n!$ with a an integer.
Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

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Irrationality measure for e, following J. Sondow

For any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbf{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!} \cdot$$

Smarandache function : $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q :

$$S(1) = 1, \; S(2) = 2, \; S(3) = 3, \; S(4) = 4, \; S(5) = 5, \; S(6) = 3 \ldots$$

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Joseph Fourier

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m \geq N+1} \frac{1}{m!}.$$

Multiply by $N!$ and set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m \geq N+1} \frac{N!}{m!},$$

so that $B_N e = A_N + R_N$. Then A_N and B_N are in $\mathbf{Z}$, $R_N > 0$ and

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In the formula

$$B_N e - A_N = R_N,$$

the numbers A_N and $B_N = N!$ are integers, while the right hand side is > 0 and tends to 0 when N tends to infinity.

Hence $N!\,e$ is not an integer, therefore e is irrational.

Since e is irrational, the same is true for $e^{1/b}$ when b is a positive integer. That e^2 is irrational is a stronger statement.

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The number e is not quadratic

Recall (Euler, 1737) : $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ which is not a periodic expansion. J.L. Lagrange (1770) : it follows that e is not a quadratic number.

Assume $ae^2 + be + c = 0$. Then

$$cN! + \sum_{n=0}^{N} (2^n a + b) \frac{N!}{n!} = -\sum_{k \geq 0} \left(2^{N+1+k} a + b\right) \frac{N!}{(N+1+k)!} \cdot$$

The left hand side is an integer, the right hand side tends to infinity. **It does not work !**

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e is not a quadratic irrationality (Liouville, 1840)

Write the quadratic equation as $ae + b + ce^{-1} = 0$.

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e is not quadratic (end of the proof)

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$$\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}=A+B+C=0$$

with

$$A=\left(a-(-1)^{N}c\right)\frac{1}{N+1}\qquad (k=0),$$

$$B=\left(a+(-1)^{N}c\right)\frac{1}{(N+1)(N+2)}\qquad (k=1),$$

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For sufficiently large N, we have $A=B=C=0$, hence $a-(-1)^N c=0$ and $a+(-1)^N c=0$, therefore $a=c=0$ and $b=0$.

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$$\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}=A+B+C=0$$

with

$$A=\left(a-(-1)^N c\right)\frac{1}{N+1} \qquad (k=0),$$

$$B=\left(a+(-1)^N c\right)\frac{1}{(N+1)(N+2)} \qquad (k=1),$$

$$C=\sum_{k\geq 2}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}.$$

For sufficiently large N, we have $A=B=C=0$, hence $a-(-1)^N c=0$ and $a+(-1)^N c=0$, therefore $a=c=0$ and $b=0$.

The number e^2 is not quadratic

J. Liouville (1809 - 1882) proved that e^2 is not a quadratic irrational number in 1840.

Sur l'irrationalité du nombre $e = 2,718\ldots$,
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e^2 is not quadratic, following Liouville

Write $ae^2 + b + ce^{-2} = 0$ and

$$\frac{N!b}{2^{N-1}} + \sum_{n=0}^{N}(a+(-1)^n c)\,\frac{N!}{2^{N-n-1}n!}$$
$$= -\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{2^k N!}{(N+1+k)!}.$$

It suffices now to check that the numbers

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Limit of the method

It does not seem that this argument will suffice to prove the irrationality of e^3, even less to prove that the number e is not a cubic irrational.

D.W. Masser noticed that the preceding proofs lead to the irrationality of $\theta = e^{\sqrt{2}} + e^{-\sqrt{2}}$, hence of $e^{\sqrt{2}}$.

In the same way we obtain the irrationality of $\sqrt{2}(e^{\sqrt{2}} - e^{-\sqrt{2}})$, but one does not deduce that $e^{\sqrt{2}}$ is not a quadratic number.

One also gets the irrationality of $e^{\sqrt{3}} + e^{-\sqrt{3}}$, hence of $e^{\sqrt{3}}$.

The domain of application of this method is quite limited.

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Irrationality criterion

Let x be a real number. The following conditions are equivalent.

(i) *x is irrational.*

(ii) *For any $\epsilon > 0$, there exists $p/q \in \mathbf{Q}$ such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{\epsilon}{q} \cdot$$

(iii) *For any real number $Q > 1$, there exists an integer q in the interval $1 \le q < Q$ and there exists an integer p such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{qQ} \cdot$$

(iv) *There exist infinitely many $p/q \in \mathbf{Q}$ satisfying*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2} \cdot$$

Irrationality of $\zeta(3)$, following Apéry (1978)

There exist two sequences of rational numbers $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$ *such that* $a_n \in \mathbf{Z}$ *and* $d_n^3 b_n \in \mathbf{Z}$ *for all* $n \geq 0$ *and*

$$\lim_{n\to\infty} |2a_n\zeta(3) - b_n|^{1/n} = (\sqrt{2}-1)^4,$$

where d_n *is the lcm of* $1, 2, \ldots, n$.

We have $d_n = e^{n+o(n)}$ and $e^3(\sqrt{2}-1)^4 < 1$.

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Idea of Ch. Hermite

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approximate the exponential function e^z by rational fractions $A(z)/B(z)$.

For proving the irrationality of e^a, (a an integer ≥ 2), approximate e^a par $A(a)/B(a)$.

If the function $B(z)e^z - A(z)$ has a zero of high multiplicity at the origin, then this function has a small modulus near 0, hence at $z = a$. Therefore $|B(a)e^a - A(a)|$ is small.

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A complex function f which is analytic near the origin is *close* to $A(z)/B(z)$ if $B(z)f(z) - A(z)$ has a zero of high multiplicity at the origin.

Example : take for f the exponential function

$$e^z = \sum_{k \geq 0} \frac{z^k}{k!}.$$

Find $B \in \mathbf{C}[z]$ (or, better, $\mathbf{Z}[z]$) such that the Taylor expansion at the origin of $B(z)e^z$ has a big gap : $A(z)$ will be the part of the expansion before the gap, $R(z) = B(z)e^z - A(z)$ the remainder.

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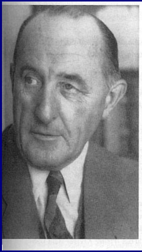
Irrationality of e^r and π

Charles Hermite (1873)

Carl Ludwig Siegel (1929, 1949)

Yuri Nesterenko (2005)



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We wish to prove the irrationality of e^a for a a positive integer, hence of e^r for $r \in \mathbf{Q}^\times$, which is equivalent to the irrationality of $\log s$ for $s \in \mathbf{Q}_{>0}$.

The same argument will give the irrationality of $\log(-1)$, meaning $\log(-1) = i\pi \not\in \mathbf{Q}(i)$, hence the irrationality of π.

Goal : write $B_n(z)e^z = A_n(z) + R_n(z)$ with A_n and B_n in $\mathbf{Z}[z]$ and $R_n(a) \neq 0$, $\lim_{n\to\infty} R_n(a) = 0$.

Substitute $z = a$, set $q = B_n(a)$, $p = A_n(a)$ and use the irrationality criterion :

$$0 < |qe^a - p| < \epsilon.$$

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Assume the number π is rational, $\pi = a/b$. Substitute $ia = i\pi b$ to z. Then $e^z = (-1)^b$, hence

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and the two complex numbers $A_n(ia)$ and $B_n(ia)$ are in $\mathbf{Z}[i]$. The left hand side is in $\mathbf{Z}[i]$, the right hand side tends to 0 as n tends to infinity. Therefore both vanish.

A resultant shows that R_n and R_{n+1} have no common zero apart from 0. Hence the contradiction.

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Rational approximation to exp

Given $n_0 \geq 0$, $n_1 \geq 0$, *find* A *and* B *in* $\mathbf{R}[z]$ *of degrees* $\leq n_0$ *and* $\leq n_1$ *such that* $R(z) = B(z)e^z - A(z)$ *has a zero at the origin of multiplicity* $\geq N + 1$ *with* $N = n_0 + n_1$.

Theorem *There is a non-trivial solution, it is unique with* B *monic. Moreover* A *and* B *are in* $\mathbf{Z}[z]$, A *has degree* n_0, B *has degree* n_1 *and* R *has multiplicity exactly* $N+1$ *at the origin.*

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$B(z)e^z = A(z) + R(z)$

Proof.
Let $D = d/dz$.
Since A has degree $\leq n_0$,

$$D^{n_0+1}R = D^{n_0+1}(B(z)e^z)$$

is the product of e^z with a polynomial of the same degree as the degree of B and same leading coefficient.
Since $D^{n_0+1}R(z)$ has a zero of multiplicity $\geq n_1$ at the origin, $D^{n_0+1}R = z^{n_1}e^z$. Hence R is the unique function satisfying $D^{n_0+1}R = z^{n_1}e^z$ with a zero of multiplicity $\geq n_0$ at 0.

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Siegel's algebraic point of view

C.L. Siegel, 1949.

$D = d/dz$

$D^{n_0+1}R(z) = z^{n_1}e^z$

The operator $J\varphi = \int_0^z \varphi(t)dt$,

inverse of D, satisfies

$$J^{n+1}\varphi = \int_0^z \frac{1}{n!}(z-t)^n \varphi(t)dt.$$

Hence

$$R(z) = \frac{1}{n_0!}\int_0^z (z-t)^{n_0} t^{n_1} e^t dt.$$

Also $A(z) = -(-1+D)^{-n_1-1}z^{n_0}$ and

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Simultaneous approximation and transcendence

The irrationality criterion involves rational approximation to a single real number θ.

We wish to prove transcendence results.

A complex number θ is transcendental if and only if the numbers

$$1, \theta, \theta^2, \ldots, \theta^m, \ldots$$

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$L = a_0 + a_1 x_1 + \cdots + a_m x_m$

Let $x_1, \ldots, x_m$ be real numbers and a_0, $a_1, \ldots,$ a_m rational integers, not all of which are zero. We wish to prove that the number

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is not zero. Approximate simultaneously $x_1, \ldots, x_m$ by rational numbers $b_1/b_0, \ldots, b_m/b_0$.
Let $b_0, b_1, \ldots, b_m$ be rational integers. For $1 \le k \le m$ set

$$\epsilon_k = b_0 x_k - b_k.$$

Then $b_0 L = A + R$ with

$$A = a_0 b_0 + \cdots + a_m b_m \in \mathbf{Z} \quad \text{and} \quad R = a_1 \epsilon_1 + \cdots + a_m \epsilon_m \in \mathbf{R}.$$

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Simultaneous approximation to the exponential function

Irrationality results follow from rational approximations $A/B \in \mathbf{Q}(x)$ to the exponential function e^x.

One of Hermite's ideas is to consider *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation.

Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbf{Z}[x]$. For $1 \le k \le m$ define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

Set $b_j = B_j(1)$, $0 \le j \le m$ and

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Hermite–Lindemann Theorem

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Hermite (1873) : transcendence of e.

Lindemann (1882) : transcendence of π.

Corollaries : transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, with $\log \alpha \neq 0$.

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Hermite : approximation to the functions $1, e^{\alpha_1 x}, \ldots, e^{\alpha_m x}$

Let $\alpha_1, \ldots, \alpha_m$ be pairwise distinct complex numbers and $n_0, \ldots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

Hermite constructs explicitly polynomials B_0, $B_1, \ldots,$ B_m with B_j of degree $N - n_j$ such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \leq k \leq m)$$

has a zero at the origin of multiplicity at least N.

Approximants de Padé

Henri Eugène Padé (1863 - 1953)
Approximation of complex analytic functions by rational functions.

Padé Approximants of type II

Let $f_0, \ldots, f_m$ be complex functions which are analytic near the origin and $n_0, \ldots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$.

There are two dual points of view, giving rise to the two types of *Padé Approximants*.

Padé approximants of second type : polynomials $B_0, \ldots, B_m$ with B_j having degree $\leq N - n_j$, such that each of the functions

$$B_i(z)f_j(z) - B_j(z)f_i(z) \quad (0 \leq i < j \leq m)$$

has a zero of multiplicity $> N$.

`Reference` : N.I. Feldman and Yu.V. Nesterenko, *Number Theory IV*, Transcendental Numbers, Encyclopaedia of Mathematical Sciences, **44** (1998) Chap. 2.

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Padé approximants of type I

Let $f_1, \ldots, f_m$ be complex functions which are analytic near the origin and let $n_1, \ldots, n_m$ be non-negative integers. Set $M = n_1 + \cdots + n_m$.

Padé approximants of the first type : polynomials $P_1, \ldots, P_m$ with P_j of degree $\le n_j$ such that the function

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Transcendental functions

A complex function is called transcendental if it is transcendental over the field $\mathbf{C}(z)$, which means that the functions z and $f(z)$ are algebraically independent : if $P \in \mathbf{C}[X, Y]$ is a non-zero polynomial, then the function $P\big(z, f(z)\big)$ is not 0.

Exercise. *An entire function (analytic in* $\mathbf{C}$*) is transcendental if and only if it is not a polynomial.*

Example. *The transcendental entire function* e^z *takes an algebraic value at an algebraic argument* z *only for* $z = 0$.

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Weierstrass question

Is it true that a transcendental entire function f takes usually *transcendental values at algebraic arguments ?*

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If S is a countable subset of $\mathbf{C}$ and T is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping S into T, as well as all its derivatives.

Also there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.

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Integer valued entire functions

An integer valued entire function is a function f, which is analytic in $\mathbf{C}$, and maps $\mathbf{N}$ into $\mathbf{Z}$.

Example : 2^z *is an integer valued entire function, not a polynomial.*

Question : *Are-there integer valued entire function growing slower than* 2^z *without being a polynomial ?*

Let f be a transcendental entire function in $\mathbf{C}$. For $R > 0$ set

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Pólya's proof starts by expanding the function f into a *Newton interpolation series* at the points $0, 1, 2, \ldots$:

$$f(z) = a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \cdots$$

Since $f(n)$ is an integer for all $n \geq 0$, the coefficients a_n are rational and one can bound the denominators. If f does not grow fast, one deduces that these coefficients vanish for sufficiently large n.

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Newton interpolation series

From

$$f(z) = f(\alpha_1) + (z - \alpha_1) f_1(z), \quad f_1(z) = f(\alpha_2) + (z - \alpha_2) f_2(z), \ldots$$

we deduce

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An identity due to Ch. Hermite

$$\frac{1}{x-z} = \frac{1}{x-\alpha} + \frac{z-\alpha}{x-\alpha} \cdot \frac{1}{x-z}.$$

Repeat :

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An identity due to Ch. Hermite

Inductively we deduce the next formula due to Hermite :

$$\frac{1}{x-z} = \sum_{j=0}^{n-1} \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_j)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} + \frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} \cdot \frac{1}{x-z}.$$

Newton interpolation expansion

Application. Multiply by $(1/2i\pi)f(z)$ and integrate :

$$f(z) = \sum_{j=0}^{n-1} A_j(z-\alpha_1)\cdots(z-\alpha_j) + R_n(z)$$

with

$$A_j(z) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{j+1})} \qquad (0 \le j \le n-1)$$

and

$$R_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)\cdot \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{F(x)dx}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)(x-z)}.$$

Integer valued entire function on $\mathbf{Z}[i]$

A.O. Gel'fond (1929) : growth of entire functions mapping the Gaussian integers into themselves.
Newton interpolation series at the points in $\mathbf{Z}[i]$.

An entire function f which is not a polynomial and satisfies $f(a+ib) \in \mathbf{Z}[i]$ for all $a+ib \in \mathbf{Z}[i]$ satisfies

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma.$$

F. Gramain (1981) : $\gamma = \pi/(2e)$.
This is best possible : *D.W. Masser (1980).*

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Transcendence of e^π

A.O. Gel'fond (1929).

If

$$e^\pi = 23,140\,692\,632\,779\,269\,005\,729\,086\,367\ \ldots$$

is algebraic, then the function $e^{\pi z}$ takes values in $\mathbf{Q}(i)$ when the argument z is in $\mathbf{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

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Hilbert's seventh problem

A.O. Gel'fond and Th. Schneider (1934).

Solution of Hilbert's seventh problem :

transcendence of α^β

and of $(\log \alpha_1)/(\log \alpha_2)$

for algebraic α, β, α_2 *and* α_2.

Dirichlet's box principle

Gel'fond and Schneider
use an auxiliary function,
the existence of which follows
from Dirichlet's box principle
(pigeonhole principle,
Thue-Siegel Lemma).

Auxiliary functions

C.L. Siegel (1929) :
Hermite's explicit formulae
can be replaced by
Dirichlet's box principle
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Slope inequalities in Arakelov theory

J–B. Bost (1994) :
matrices and determinants require choices of bases.
Arakelov's Theory produces *slope inequalities* which avoid the need of bases.

Périodes et isogénies des variétés abéliennes sur les corps de nombres, (d'après D. Masser et G. Wüstholz).
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Rational interpolation

René Lagrange (1935).

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha}\cdot\frac{z-\alpha}{z-\beta}\cdot\frac{1}{x-z}.$$

Iterating and integrating yield

$$f(z) = \sum_{n=0}^{N-1} B_n \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_n)} + \tilde{R}_N(z).$$

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Hurwitz zeta function

T. Rivoal (2006) : consider Hurwitz zeta function

$$\zeta(s,z)=\sum_{k=1}^{\infty}\frac{1}{(k+z)^s}.$$

Expand $\zeta(2,z)$ as a series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}.$$

The coefficients of the expansion belong to $\mathbf{Q}+\mathbf{Q}\zeta(3)$.
This produces a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.
In the same way : new proof of the irrationality of $\log 2$ by expanding

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Mixing C. Hermite and R. Lagrange

T. Rivoal (2006) : new proof of the irrationality of $\zeta(2)$ by expanding

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}.$$

Taylor series and interpolation series

Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants.

Multiplicities can also be introduced in René Lagrange interpolation.

There is another duality between the methods of Gel'fond and Schneider : Fourier-Borel transform.

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Transcendence and algebraic independence of values of modular functions (*méthode stéphanoise* and work of Yu.V. Nesterenko).

Measures : transcendence, linear independence, algebraic independence...

Finite characteristic :

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History of irrational and transcendental numbers

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/