Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§3 Spherical functions (February 20, 2009)

Taking

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K,$$

we know that $k(\theta)$ acts as the hyperbolic rotation with angle 2θ around the point i, since $k(\pi) = -I$ acts trivially. The pair (r, φ) is called the geodesic polar coordinates of the point $z \in \mathbb{H}$, where $r = \rho(z, i)$ and 2φ is the angle that the geodesic passing through z and i forms with $i\mathbb{R}$.

Recall we have given eigenfunctions f of Laplace operator Δ with eigenvalue λ, which satisfy

$$f(n(x)z) = \chi(x)f(z), \quad \text{for all } n \in N.$$

Here $\chi : N \to \mathbb{C}$ is the character given by

$$\chi(n) = e(x), \quad \text{if } n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

We assume

$$f(z) = e(x)F(2\pi y).$$

Specifically, we get the Whittaker function

$$W_s(z) = 2\sqrt{y}K_{s-\frac{1}{2}}(2\pi y)e(x),$$

where $K(y)$ is Bessel function satisfying

$$K(y) \sim e^{-y}, \quad \text{as } y \to \infty.$$

In fact, by changing variable $z \mapsto rz = \begin{pmatrix} \sqrt{r} & \\ & \frac{1}{\sqrt{r}} \end{pmatrix} z$, $W_s(rz)$ is also an eigenfunction of Δ with the same eigenvalue, and satisfies

$$W_s(rz) = e(rx)F(2\pi ry),$$

since Δ commutes with the action of G.

Proposition. *Any $f \in \mathcal{C}_c^\infty(\mathbb{H})$ has the integral representation*

$$f(z) = \frac{1}{2\pi i}\int_{\Re s=\frac{1}{2}}\int_{\mathbb{R}} W_s(rz)f_s(r)\gamma_s(r)drds,$$

where the outer integration is taken over the vertical line $s = 1/2 + it$,

$$f_s(r) = \Big(f, W_s(r)\Big) = \int_{\mathbb{H}} f(z)W_s(rz)d\mu z,$$

and $\gamma_s(r) = (2\pi|r|)^{-1}t\sinh\pi t$.

The proof is based on Fourier inversion formula in (r, x) variable and the following Kontorovitch-Lebedev inversion in (t, y) variables:

$$G(x) = \int_0^{+\infty} K_{ix}(y) g(y) y^{-1} dy,$$

then

$$g(x) = \pi^{-2} \int_0^{\infty} K_{ix}(r) G(t) t \sinh(\pi t) dt.$$

Proposition. *Let $f(z)$ be an eigenfunction of Δ with eigenvalue $\lambda = s(1-s)$ which satisfies the following transformation*

$$f(z+m) = f(z) \quad \textit{for all } m \in \mathbb{Z}$$

and the growth condition

$$f(z) = o(e^{2\pi y}) \quad \textit{as } y \to +\infty.$$

Then $f(z)$ has the expansion

$$f(z) = f_0(y) + \sum_{n \neq 0} f_n W_s(nz),$$

where $f_n = (f, W_s(n))_{\Gamma_\infty \backslash \mathbb{H}}$ and f_0 is a linear combination of y^s and y^{1-s} if $\lambda \neq \frac{1}{4}$; or $y^{\frac{1}{2}}$ and $y^{\frac{1}{2}} \log y$ if $\lambda = \frac{1}{4}$.

The proof follows from usual Fourier series expansion for periodic function and the uniqueness of rapidly decreasing solution.

We can get similar solutions in polar coordinates. Consider solutions of $(\Delta + \lambda) f = 0$ which satisfy

$$f(kz) = \chi_m(k) f(z), \quad \text{for all } k \in K,$$

where $\chi : K \to \mathbb{C}$ is the character given by (for $m \in \mathbb{Z}$)

$$\chi_m(k(\theta)) = e^{2im\theta}, \quad \text{if } k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

With respect to $z = k(\varphi) e^{-r} i$, we can take

$$\begin{aligned} f(z) &= \frac{1}{\pi} \int_0^{\pi} (\mathrm{Im} k(\theta) z)^s \overline{\chi(k)} d\theta \\ &= \frac{\Gamma(1-s)}{\Gamma(1-s+m)} P_{-s}^m(\cosh r) e^{2im\varphi}, \end{aligned}$$

where $P_{-s}^m(v)$ is the Legendre function.

We give the definition of the classical spherical function as

$$U_s^m(z) = P_{-s}^m(\cosh r) e^{2im\varphi} = P_{-s}^m(2u+1) e^{2im\varphi}.$$

We have a similar expansion for any $f \in \mathcal{C}_c^\infty(\mathbb{H})$.

Proposition. *Any $f \in \mathcal{C}_c^\infty(\mathbb{H})$ has the expansion*

$$f(z) = \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{2\pi i} \int_{\Re s=1/2} U_s^m(z) f^m(s)\delta(s)ds,$$

where

$$f^m(s) = \big(f, U_s^m\big) = \int_{\mathbb{H}} f(z)U_s^m(z)d\mu z, \qquad \delta(s) = t\tanh \pi t.$$

The proof is based on expanding the Fourier series in φ with the following inversion formula due to F. G. Mehler and V. A. Fock. If

$$G(t) = \int_0^{+\infty} P_{-\frac{1}{2}+it}(y)g(y)\frac{dy}{y},$$

then

$$g(u) = \int_0^\infty P_{-\frac{1}{2}+it}(u)G(t)t\tanh(\pi t)dt.$$

Here $P_s(v) = P_s^0(v)$ denotes the Legendre function of order $m = 0$.

The spherical functions of order zero are special; they depend only on the hyperbolic distance $\rho(z,i)$.

Theorem. *For any λ, there is a unique spherical function of Δ which satisfies $f(i) = 1$ and $(\Delta + \lambda)f = 0$.*

Proof. We have $f(z) = F(u)$ since $f(z)$ is spherical, where $\cosh r = 1 + 2u$. With respect to the polar coordinates, $F(u)$ satisfies a 2^{nd} order differential equation in u, i.e.

$$u(u+1)F''(u) + (2u+1)F' + s(1-s)F = 0.$$

This has a unique solution with $F(0) = 1$, since the other one has singularity at 0. $\square$

This is important. We will want to study the eigenfunctions of Δ subject to additional conditions. It is better sometimes to replace differential operators by integral operators (equations).

What kind of integral operators? Consider a general integral operator on $\mathbb{H}$ which is defined by

$$(L_k f)(z) = \int_{\mathbb{H}} k(z,w)f(w)d\mu(w).$$

Here $d\mu$ is the Riemannian measure and $k : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$ is a given function, called the kernel of L_k. Define $T_g f(z) = f(gz)$. The next question is when L_k commutes with all these T_g, $\forall$ $g \in G$? In fact, we have

$$\begin{aligned} L_k(T_g f) = T_g(L_k f) \quad &\Leftrightarrow \quad \int_{\mathbb{H}} k(gz,w)f(w)d\mu(w) = \int_{\mathbb{H}} k(z,g^{-1}w)f(w)d\mu(w) \\ &\Leftrightarrow \quad k(gz,gw) = k(z,w) \end{aligned}$$

It is necessary and sufficient that $k(z, w)$ depends only on $\rho(z, w)$. A function with this property is called point-pair invariant.

Remark. In the Euclidean case, the G-invariant integral operator is just the convolution operator

$$(F * f)(x) = \int_{\mathbb{R}} F(x-y) f(y) \mathrm{d}y,$$

where $F(x-y)$ is the kernel function.

Back to $\mathbb{H}$, we usually write

$$k = k(u(z, w)),$$

where $k(u)$ is a function in one variable $u \geq 0$ and $u(z, w)$ is given in §2. As usual, $\cosh \rho(z, w) = 1 + 2u(z, w)$.

Theorem. *Any eigenfunction of $\triangle$ is also an eigenfunction of the G-invariant integral L_k. In fact, if $(\triangle + \lambda) f = 0$, then*

$$\int_{\mathbb{H}} k(u(z, w)) f(w) \mathrm{d}\mu(w) = \Lambda f(z),$$

where $\Lambda = \Lambda(\lambda, k) \in \mathbb{C}$, depending on λ and k but not on f.

Conversely, if f is a eigenfunction of all L_k (k is point-pair invariant), then f is $\triangle$-eigenfunction.

Proof. What the Theorem basically says is that L_k is in the algebra generated by $\triangle$ in a suitable sense, or a function of $\triangle$.

$$F(\triangle) f + F(\lambda) f = 0 \quad \text{for any } F.$$

To prove the theorem, We shall give a proposition firstly.

Proposition. *There exits a unique function $\omega(z, w)$ satisfies the following three condition:*

i) For any fixed w, $z \mapsto \omega(z, w)$ depends only on $\rho(z, w)$;

ii) $(\Delta_z + \lambda)\omega(z, w) = 0$;

iii) $\omega(w, w) = 1$.

In fact $\omega(z, w) = F_s(u(z, w))$, where F_s is a Legendre function.

Proof. We have proved it for $w = i$. If $w = gi$, then

$$\omega(z, w) = \omega(g^{-1}z, i) = F_s(u(g^{-1}z, i)) = F_s(u(gz, w)). \ \square$$

We come back to the proof. Suppose that $(\triangle + \lambda) f = 0$. Fix $w \in \mathbb{H}$ and introduce mean value operator at w by

$$f_w(z) = \int_{G_w} f(gz) \mathrm{d}g,$$

where $G_w = \mathrm{Stab}(w) = h^{-1}Kh$, if $w = hi$. And $\mathrm{d}g$ is a Haar measure in G_w normalized by $\mathrm{vol}(G_w) = 1$.

We have $(\triangle+\lambda)f_w = 0$, because $\triangle$ commutes with G. $f_w(z)$ depends only on $\rho(z,w)$ and $f_w(gz) = f_w(z)$, $\forall g \in G_w$. By the proposition, we obtain

$$f_w(z) = f_w(w)\omega(z,w) = f(w)\omega(z,w), \tag{0.1}$$

because

$$f_w(w) = \int_{G_w} f(gw)\mathrm{d}g = \int_{G_w} f(w)\mathrm{d}g = f(w).$$

Now we claim that

$$(Lf)(z) = (Lf_z)(z).$$

Indeed, we have

$$\begin{aligned}
(Lf_z)(z) &= \int_{\mathbb{H}} k(z,w)f_z(w)\mathrm{d}\mu(w) = \int_{\mathbb{H}}\int_{G_z} k(z,w)f(gw)\mathrm{d}g\mathrm{d}\mu(w) \\
&= \int_{G_z}\int_{\mathbb{H}} k(z,w)f(gw)\mathrm{d}\mu(w)\mathrm{d}g \xlongequal{\text{chang } w \text{ to } g^{-1}w} \int_{G_z}\int_{\mathbb{H}} k(z,g^{-1}w)f(w)\mathrm{d}\mu(w)\mathrm{d}g \\
&\xlongequal{k \text{ point-pair invariant}} \int_{G_z}\int_{\mathbb{H}} k(gz,w)f(w)\mathrm{d}\mu(w)\mathrm{d}g = \int_{\mathbb{H}}\int_{G_z} k(gz,w)f(w)\mathrm{d}\mu(w)\mathrm{d}g \\
&= \int_{\mathbb{H}}\int_{G_z} k(z,w)f(w)\mathrm{d}\mu(w)\mathrm{d}g = \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w) \\
&= (Lf)(z).
\end{aligned}$$

Going back and using (0.1), we get

$$\begin{aligned}
(Lf)(z) &= (Lf_z)(z) = L\left(w \mapsto f(z)\omega(z,w)\right)(z) \\
&= f(z)L(\omega(z,w))(z),
\end{aligned}$$

where ω is symmetric and

$$L(\omega(z,w))(z) = \Lambda_{k,\lambda} = \int_{\mathbb{H}} F_s(u(z,w))k((z,w))\mathrm{d}\mu(w).$$

We still have another part of this theorem. Note that L commutes with $\triangle$. Indeed, we have

$$\begin{aligned}
L(\triangle f)(z) &= \int_{\mathbb{H}} k(z,w)(\triangle f)(w)\mathrm{d}\mu(w) \\
&= \int_{\mathbb{H}} \triangle_w k(z,w)f(w)\mathrm{d}\mu(w) \\
&= \int_{\mathbb{H}} \triangle_z k(z,w)f(w)\mathrm{d}\mu(w) \\
&= \triangle\left(z \mapsto \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w)\right) \\
&= (\triangle Lf)(z).
\end{aligned}$$

Because

$$\triangle_w k(z,w) = \triangle_z k(z,w).$$

To check it, using polar coordinates, one can check that

$$\begin{aligned}\triangle_z k(z,w) &= u(u+1)k''(u) + (2u+1)k'(u)\\ &= \triangle_w k(z,w).\end{aligned}$$

This deduces that $\triangle_z k(z,w)$ is a point-pair invariant function.

Converse direction: Suppose that f is an eigenfunction of L_k for all k point-pair invariant. Then we show that f is a eigenfunction of $\triangle$. Without loss of generality, we assume $f(i) \neq 0$, otherwise, we can translate f. Then we take k, s. t.

$$\int k(i,w)f(w)\mathrm{d}\mu(w) \neq 0,$$

where $k(0) = 1$, k has small support near 0.

By assumption,

$$(Lf)(z) = \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w) = \Lambda f(z),\ \Lambda \neq 0.$$

Simply apply $\triangle$ to the above equation, we get

$$L\triangle f = \triangle L f = \Lambda \triangle f,$$

since $\triangle$ and L commutes. This deduces that $\triangle f$ is also an eigenfunction of L.

On the other hand, we have

$$\triangle L f = \int_{\mathbb{H}} \triangle_z k(z,w)f(w)\mathrm{d}\mu(w),$$

where $\triangle_z k(z,w)$ is also a point-pair invariant function.

By assumption, we have

$$L_{\tilde{k}} f = \Lambda' f.$$

Then we get

$$\triangle f = \frac{L\triangle f}{\Lambda} = \frac{\triangle L f}{\Lambda} = \frac{\Lambda' f}{\Lambda}.$$

Remarks.

(1) How to obtain $\Lambda(k,\lambda)$? The map

$$k \mapsto h(s) = \Lambda(k,\lambda) = \int \omega(z,w)h(z,w)\mathrm{d}\mu(w),$$

where $\lambda = s(1-s)$. In fact,

$$h(t) = 4\pi \int_0^\infty F_s(u)k(u)\mathrm{d}u.$$

The inversion formula is given by,

$$k(u) = \frac{1}{4\pi}\int_0^\infty F_s(u)h(t)t\tanh \pi t \mathrm{d}t.$$

(2) From the expansion of $f \in \mathcal{C}_c^\infty(\mathbb{H})$ in terms of W_s, $\Re s = \frac{1}{2}$, we deduce that

$$\operatorname{Spec}(\triangle) \subseteq (-\infty, -\frac{1}{4}],$$

which is equivalent to

$$\frac{1}{4} \parallel f \parallel_2 \leq \parallel \triangle f \parallel_2 .$$

One can see it directly, (I add: there is something wrong to a square!)

Let $f \in \mathcal{C}_c^\infty(\mathbb{H})$, then we have

$$\begin{aligned}\int_0^\infty f^2(x+iy)\frac{\mathrm{d}y}{y^2} &= 2\int_0^\infty f(x+iy)\frac{\partial f}{\partial y}(x+iy)\frac{\mathrm{d}y}{y} \\ &\leq 2\left(\int_0^\infty \left(\frac{\partial f}{\partial y}\right)^2 (x+iy)\mathrm{d}y\right)^{1/2}\left(\int_0^\infty f^2(x+iy)\frac{\mathrm{d}y}{y^2}\right)^{1/2},\end{aligned}$$

and then

$$\begin{aligned}\frac{1}{4}\int_0^\infty f^2(x+iy)\frac{\mathrm{d}y}{y^2} &\leq \int_0^\infty \left(\frac{\partial f}{\partial y}\right)^2 (x+iy)\mathrm{d}y \\ &\leq \int_0^\infty \left|\left(\frac{\partial f}{\partial x}\right)^2 (x+iy) + \left(\frac{\partial f}{\partial y}\right)^2 (x+iy)\right| \mathrm{d}y \\ &= \int_0^\infty |\triangle f|^2 \frac{\mathrm{d}y}{y^2}.\end{aligned}$$

We can get

$$(\triangle f, f) = \int_0^\infty \left|\frac{\partial^2 f}{\partial x^2}\right| \mathrm{d}x.$$

Integrating over x, we have

$$\frac{1}{4}\int_{\mathbb{H}} f^2(z)\mathrm{d}\mu(z) \leq \int_{\mathbb{H}} (\triangle f)^2(z)\mathrm{d}\mu(z).$$

An attractive way to get $k \mapsto h$ is using y^s, $\lambda = s(1-s)$ in the following three steps. We have

$$q(v) = \int_v^\infty k(u)(u-v)^{-1/2}\mathrm{d}u.$$

By Abel transform, we get

$$g(r) = 2q\left(\sinh(r/2)^2\right),$$

and

$$h(t) = \int_{\mathbb{R}} e^{irt} g(r)\mathrm{d}r.$$

References

[1] V. A. Fock, *On the representation of an arbitrary function by an integral involving Legendre's function with a complex index.* C. R. (Dokl.) Acad. Sci. URSs 39 (1943), 253-256.