Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§7 Selberg Trace Formula (March 2, 2009)

In 1956, A. Selberg derived the trace formula from the spectral theorem, which established a quantitative connection between the spectrum and the geometry of the Riemaan surface. A simple case is:

$$\Gamma\backslash\mathbb{H} \text{ is compact} \Longleftrightarrow \Gamma\backslash G \text{ is compact.}$$

Let L_k be the invariant operators, such that

$$Lf(z) = \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w), \tag{0.1}$$

where

$$k(z,w) = k(u(z,w)), \quad \text{and} \quad u(z,w) = \frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}.$$

We have

$$\cosh\rho(z,w) = 1 + 2u(z,w).$$

$k(z,w)$ is a point-pair invariant function.

Denote

$$L : L^2(\Gamma\backslash\mathbb{H}) \to L^2(\Gamma\backslash\mathbb{H})$$

be the integral operator with kernel $K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)$.

Theorem. *If φ is an eigenfunction of Δ with eigenvalue $\lambda = s(1-s) = \frac{1}{4} + t^2$, where $s = \frac{1}{2} + it$, $t \in \mathbb{C}$, then $L\varphi = h(t)\varphi$.*

We need an explicit expression for h in terms of the eigenvalue λ and the kernel function $k(u)$. This is given by the Selberg transform in the following three steps:

$$q(v) = \int_0^\infty k(u)(u-v)^{-1/2}\mathrm{d}u,$$

$$g(r) = 2q\Big(\big(\sinh\frac{r}{2}\big)^2\Big),$$

$$h(t) = \int_{-\infty}^{+\infty} e^{irt}g(r)\mathrm{d}r.$$

Since $\Gamma\backslash\mathbb{H}$ is compact, then L is a trace-class operator on $L^2(\Gamma\backslash\mathbb{H})$, and

$$\mathrm{Tr}L = \int_{\Gamma\backslash\mathbb{H}} K(z,z)\mathrm{d}\mu(z).$$

We know Δ has discrete spectrum on $\Gamma\backslash\mathbb{H}$, so does L. We can write down

$$K(z,w) = \sum_{j=0}^{\infty} h(t_j)u_j(z)\overline{u_j(w)},$$

where $\{u_j\}_{j=0}^{\infty}$ is the orthonormal basis of eigenfunctions of Δ in the space $L^2(\Gamma\backslash\mathbb{H})$.

If we denote

$$(\Delta + \lambda_j)u_j = 0, \quad \text{where} \quad \lambda_j = \frac{1}{4} + t_j^2,$$

we obtain

$$\int_{\Gamma\backslash\mathbb{H}} K(z,z)\mathrm{d}\mu(z) = \sum_j h(t_j) \int_{\Gamma\backslash\mathbb{H}} |u_j(z)|^2 \mathrm{d}\mu(z) = \sum_j h(t_j).$$

Question. Can we write the trace in a different way?
Following Selberg, we partition the group Γ into conjugacy classes. Given the conjugacy classes $\mathcal{C}$ in Γ, then we have $\Gamma = \coprod \mathcal{C}$ and we can also write

$$K(z,z) = \sum_{\gamma\in\Gamma} k(z,\gamma z) = \sum_{\mathcal{C}} \sum_{\gamma\in\mathcal{C}} k(z,\gamma z) = \sum_{\mathcal{C}} \widetilde{K}_{\mathcal{C}},$$

where $\widetilde{K}_{\mathcal{C}}(z) = \sum_{\gamma\in\mathcal{C}} k(z,\gamma z)$ is a Γ-invariant function of z. On the other hand, we get

$$\int_{\Gamma\backslash\mathbb{H}} K(z,z)\mathrm{d}\mu(z) = \sum_{\mathcal{C}} \int_{\Gamma\backslash\mathbb{H}} \widetilde{K}_{\mathcal{C}}(z)\mathrm{d}\mu(z).$$

Question. How to compute the summands?
We define the conjugacy class $\mathcal{C} = [\gamma] = \{\tau^{-1}\gamma\tau : \tau\in\Gamma\}$ and the centralizer $\mathcal{C}_\Gamma(\gamma) = \{\rho \in \Gamma : \rho\gamma = \gamma\rho\}$. Take $\mathcal{C} \leftrightarrow C_\Gamma(\gamma)\backslash\Gamma$: $\gamma \mapsto \tau^{-1}\gamma\tau$ and choose $\gamma\in\mathcal{C}$, we can write

$$\widetilde{K}_{\mathcal{C}}(z) = \sum_{\tau\in\mathcal{C}_\Gamma(\gamma)\backslash\Gamma} k(z,\tau^{-1}\gamma\tau z) = \sum_{\tau\in\mathcal{C}_\Gamma(\gamma)\backslash\Gamma} k(\tau z,\gamma\tau z).$$

Therefore

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} \widetilde{K}_{\mathcal{C}}(z)\mathrm{d}\mu z &= \int_{\Gamma\backslash\mathbb{H}} \sum_{\tau\in\mathcal{C}_\Gamma(\gamma)\backslash\Gamma} k(\tau z,\gamma\tau z)\mathrm{d}\mu z \\ &= \int_{\mathcal{C}_\Gamma(\gamma)\backslash\mathbb{H}} k(z,\gamma z)\mathrm{d}\mu(z).\end{aligned}$$

Observe that this expression depends only on τ and not on γ itself. It is simple because $\mathcal{C}_\Gamma(\gamma)$ is simple, and the fundamental domain of $\mathcal{C}_\Gamma(\gamma)$ in $\mathbb{H}$ is well-understood. Each element in G is conjugate to either N, A or K, where

$$\begin{aligned}N &= \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} : x\in\mathbb{R}\right\},\\ A &= \left\{\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix} : a>0\right\},\\ K &= \left\{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} : \theta\in\mathbb{R}\right\}.\end{aligned}$$

If $\gamma\neq I$, $\mathcal{C}_\Gamma(\gamma) \subseteq \{$stabilizer in Γ of the fixed points of $\gamma\in\widehat{\mathbb{C}}\}$, the stabilizer is a cyclic group.

The fundamental domain $C_\Gamma(\gamma)\backslash\mathbb{H}$ is as simple as a vertical strip, a horizontal strip or a sector in $\mathbb{H}$, after γ is brought to N, A, K, respectively, by conjugation, we have following pictures

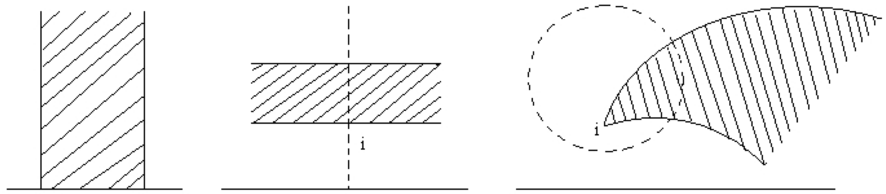

FIGURE 1

Since Γ is co-compact, it does not contain parabolic elements, so only the last two possibilities can occur.

Computing the trace for the identity motion.

For $\mathcal{C} = 1$, we have $\tilde{K}_{\mathcal{C}}(z) = k(z,z)$ and

$$\int_{\Gamma\backslash\mathbb{H}} \widetilde{K}_{\mathcal{C}}(z)\mathrm{d}\mu z = \int_{\Gamma\backslash\mathbb{H}} k(z,z)\mathrm{d}\mu z = k(0)\mathrm{area}(\Gamma\backslash\mathbb{H}),$$

where

$$k(0) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} r\tanh(\pi r)h(r)\mathrm{d}r.$$

We would like to have an identity involving g, h (Fourier transform of each other).

Computing the trace for hyperbolic classes.

Now we compute in the case: γ is hyperbolic.
We denote the primitive hyperbolic conjugacy classes in Γ by P, which correspond to closed geodesics. Let $\mathcal{C} = P^l$, choose $\gamma = \gamma_P^l$, $l \in \mathbb{Z}$ and $\gamma_P \in P$ is primitive(γ_P generates the stabilizer in Γ of its fixed points in $\widehat{\mathbb{C}}$). Then $\mathcal{C}_\Gamma(\gamma) = \mathcal{C}_\Gamma(\gamma_P)$. By conjugation, $\gamma_p \sim \begin{pmatrix}\sqrt{p} & \\ & \sqrt{p}^{-1}\end{pmatrix}$. Then $\log p$ is the hyperbolic distance of i to pi, thus also the distance of z to $\gamma_P z$ for any z on the geodesic joining the fixed points of γ_P. This geodesic closes on the surface $\Gamma\backslash\mathbb{H}$. The length of the geodesic in $\Gamma\backslash\mathbb{H} = \dfrac{\log p}{\text{winding }\sharp}$. Its fundamental domain is the horizontal strip $1 < y < p$. Hence we get

$$\int_{C_\Gamma(\gamma)\backslash\mathbb{H}} k(z,\gamma z)\mathrm{d}\mu(z) = \int_1^p\int_{-\infty}^{+\infty} k(z,p^l z)\mathrm{d}\mu(z)$$

for $\gamma = \begin{pmatrix} \sqrt{p}^{\,l} & \\ & \sqrt{p}^{\,-l} \end{pmatrix}$. Putting $2d = |p^{l/2} - p^{-l/2}|(d = \sinh p^{l/2})$, we continue the computation as follows:

$$\begin{aligned}
\int_1^p \int_{-\infty}^{+\infty} k(z, p^l z)\mathrm{d}\mu(z) &= \int_1^p \int_{-\infty}^{+\infty} k\left(\left(\frac{d|z|}{y}\right)^2\right) \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&\overset{x\mapsto xy}{=\!=\!=} \int_1^p \frac{\mathrm{d}y}{y} \int_{-\infty}^{+\infty} k(d^2(x^2+1))\mathrm{d}x \\
&\overset{u=d^2(x^2+1)}{=\!=\!=\!=\!=} \frac{\log p}{d} \int_{d^2}^{+\infty} \frac{k(u)}{\sqrt{u-d^2}}\mathrm{d}u = \frac{\log p}{d} q(d^2) \\
&= \frac{\log p}{2d} g\left(2\log(\sqrt{d^2+1}+d)\right) = \frac{\log p}{2d} g(l\log p) \\
&= |p^{l/2} - p^{-l/2}|^{-1} g(l \log p)\log p.
\end{aligned}$$

If Γ has no torsion, then we have done. Otherwise, there are finitely many elliptic conjugacy classes.

Computing the trace for elliptic classes.

Denote by $\mathcal{R}$ a primitive elliptic conjugacy class (rotation with minimum angle) in Γ of order m. Any elliptic class having the same fixed points as $\mathcal{R}$ is $\mathcal{C} = \mathcal{R}^l$ with $0 < l < m$. Conjugating $\mathcal{R}$ in G, we can assume the representative to be $k(\theta)$, this acts as a rotation of angle 2θ around i, where $\theta = \frac{\pi}{m}$. Since it generates the centralizer, i. e. $\langle k(\theta)\rangle = \mathcal{C}_\Gamma(\gamma)$, the fundamental domain $\mathcal{S}$ of that centralizer is a hyperbolic sector of angle 2θ.

Therefore,

$$\text{our integral is } = \int_{\mathcal{S}} k\big(z, k(\theta l)z\big)\mathrm{d}\mu(z) = \frac{1}{m}\int_{\mathbb{H}} k\big(z, k(\theta l)z\big)\mathrm{d}\mu(z), \tag{0.2}$$

because it takes m images of $\mathcal{S}$ to cover $\mathbb{H}$ exactly (except for a zero measure set).

We shall continue computation using geodesic polar coordinates $z = k(\varphi)e^{-r}i$, where φ ranges over $[0, \pi)$ and r over $[0, +\infty)$. Since $k(\varphi)$ commutes with $k(\theta l)$, we get

$$k\big(k(\varphi)z, k(\theta l)k(\varphi)z\big) = k\big(z, k(\theta l)z\big),$$

which is a function depends only on variable r. (0.2) becomes

$$\frac{\pi}{m}\int_0^\infty k\big(e^{-r}i, k(\theta l)e^{-r}i\big)(2\sinh r)\mathrm{d}r.$$

By $u\big(z, k(\theta)z\big) = \dfrac{|z^2+1|^2\sin^2\theta}{(2y)^2} = (\sinh r \sin\theta)^2$ if $z = e^{-r}i$, we get

$$\begin{aligned}
\mathrm{Tr}L &= \frac{\pi}{m}\int_0^\infty k\big((\sinh r\sin\theta l)^2\big)(2\sinh r)\mathrm{d}r \\
&\overset{u=\sinh r\sin\theta l}{=\!=\!=\!=\!=} \frac{\pi}{m\sin\theta l}\int_0^\infty \frac{k(u)\mathrm{d}u}{\sqrt{u^2+\sin^2\theta l}}.
\end{aligned}$$

Let $\sigma = \sin\theta l$, we get

$$\int_0^\infty \frac{k(u)\mathrm{d}u}{\sqrt{u^2+\sigma^2}} \xlongequal{\text{integration by parts}} -\frac{1}{\pi}\int_0^\infty q'(v)\int_0^v \frac{1}{\sqrt{(v-u)(u^2+\sigma^2)}}\mathrm{d}u\mathrm{d}v$$

$$\xlongequal{\text{linear change of variable in } u} -\frac{1}{\pi}\int_0^\infty q'(v)\int_0^{v/(v+\sigma^2)} \frac{\mathrm{d}u}{\sqrt{u(1-u)}}\mathrm{d}v$$

$$\xlongequal{\text{integration by parts}} \frac{\sigma}{\pi}\int_0^\infty q(v)\frac{\mathrm{d}u}{(v+\sigma^2)\sqrt{v}}\mathrm{d}v$$

$$\xlongequal{\text{by changing } v=\sinh^2(r/2)} \frac{\sigma}{2\pi}\int_0^\infty \frac{g(r)\cosh(r/2)}{\sinh^2(r/2)+\sigma^2}\mathrm{d}r,$$

in which

$$q(v) = \int_v^\infty \frac{k(u)\mathrm{d}u}{\sqrt{v-u}},$$

and

$$g(r) = 2q\left(\sinh(r/2)^2\right).$$

For $\sigma = \sin\alpha > 0$ this yields

$$\int_0^\infty \frac{k(u)\mathrm{d}u}{\sqrt{u^2+\sin\alpha^2}} = \frac{\sin\alpha}{\pi}\int_0^\infty \frac{g(r)\cosh(r/2)}{\cosh r - \cos 2\alpha}\mathrm{d}r,$$

where $\alpha = \frac{\pi l}{m}$.

If one wants to have an expression in terms of h rather than g, we use

$$\int_0^\infty g(r)f(r)\mathrm{d}r = \int_0^\infty h(r)\hat{f}(r)\mathrm{d}r.$$

So we need to compute Fourier transform of f.

Ex. 1 The Fourier transform of f is

$$\frac{1}{2\sin\alpha}\frac{\cosh(\pi-2\alpha)r}{\cosh\pi r}.$$

By Ex 1 we get

$$\mathrm{Tr}L = \frac{1}{2m\sin(\pi l/m)}\int_{-\infty}^{+\infty} h(r)\frac{\cosh\pi(1-2l/m)r}{\cosh\pi r}\mathrm{d}r.$$

This finishes the story for the co-compact case, now uniform case ($\Gamma = SL_2(\mathbb{Z})$). There are several problems:

(1) L (as in the integral operator) with kernel $k(z,w)$ is not of trace class.

(2) $\int_{\Gamma\backslash\mathbb{H}} K(z,z)\mathrm{d}\mu(z)$ does not converge.

(3) One has continuous spectrum. Need regularization.

Naive way

$$\int_{\mathcal{F}(T)} K(z,z)\mathrm{d}\mu(z)$$

It turns out that it is

$$A \log T + B + o(1), \quad \text{as } T \to \infty.$$

Use truncation, for any φ on $\mathbb{H}$

$$\Lambda^T \varphi = \varphi - \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_p(\mathrm{Im}\gamma z) \chi_{>T}(\mathrm{Im}\gamma z),$$

where $\Lambda^T \varphi$ is Γ-invariant, and $\chi_{>T}$ is the character function of (T, ∞).

For $z \in \mathcal{F}$,

$$\Lambda_z^T \varphi(z) = \begin{cases} \varphi(z), & \mathrm{Im}z \leq T, \\ \varphi(z) - \varphi_P(y), & \mathrm{Im}z > T, \end{cases}$$

We will compute

$$\int_{\Gamma \backslash \mathbb{H}} \Lambda_z^T K(z, w) \Big|_{w=z} \mathrm{d}\mu(z),$$

denote the integrated function by $K^T(z)$.

Spectral side: Starting point

$$K(z, w) = \sum h(t_j) u_j(z) \overline{u_j(w)} + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(r) E(z; \frac{1}{2} + ir) \overline{E(w, \frac{1}{2} + ir)} \mathrm{d}r,$$

where $u_0 = \dfrac{1}{\sqrt{\mathrm{area}(\Gamma \backslash \mathbb{H})}}$, u_j are cusp forms, $j > 0$, and $\lambda_j = \frac{1}{4} + t_j^2$, $t_0 = \frac{i}{2}$.

$$\int_{\Gamma \backslash \mathbb{H}} K^T(z) \mathrm{d}\mu(z) = h(\frac{i}{2}) \frac{\mathrm{area}(\mathcal{F}(T))}{\mathrm{area}(\Gamma \backslash \mathbb{H})} + \sum_{j=1}^{\infty} h(t_j)$$
$$+ \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(r) \parallel \Lambda^T E(z; \frac{1}{2} + it) \parallel^2,$$

where $\Lambda^T u_j = u_j$ for $j > 0$.

Ex 2. Prove

$$\Lambda^T 1 = \chi_{\mathcal{F}(T)}$$

Ex 3. Prove

$$(\Lambda^T \varphi_1, \varphi_2) = (\varphi_1, \Lambda^T \varphi_2)$$

By Maass-Selberg relations,

$$\parallel \Lambda^T E(z; \frac{1}{2} + it) \parallel^2 = 2 \log T - \frac{\phi'}{\phi}(\frac{1}{2} + ir) + \frac{\phi(\frac{1}{2} - ir) T^{2ir} - \phi(\frac{1}{2} + ir) T^{-2ir}}{2ir},$$

where

$$\phi(s) = \frac{\zeta^*(2s - 1)}{\zeta^*(2s)}.$$

Hence we need to compute

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{-\infty}^{+\infty} h(r)\frac{\phi(\frac{1}{2}-ir)T^{2ir}-\phi(\frac{1}{2}+ir)T^{-2ir}}{2ir}\mathrm{d}r \\
=&\frac{1}{4\pi i}\int_{-\infty}^{+\infty} h(r)\frac{\phi(\frac{1}{2}-ir)T^{2ir}-\phi(\frac{1}{2})}{r}\mathrm{d}r \\
=&\frac{1}{4\pi i}\int_{-\infty}^{+\infty} h(r)\frac{\phi(\frac{1}{2}-ir)-\phi(\frac{1}{2})}{2ir}T^{2ir}\mathrm{d}r \\
&-\frac{1}{4\pi i}\phi(\frac{1}{2})\int_{-\infty}^{+\infty} h(r)\frac{T^{2ir-1}}{r}\mathrm{d}r \\
=&\frac{1}{4\pi i}\int_{-\infty}^{+\infty} h(r)\frac{\phi(\frac{1}{2}-ir)-\phi(\frac{1}{2})}{2ir}T^{2ir}\mathrm{d}r \\
&-\frac{1}{4\pi i}\phi(\frac{1}{2})\int_{-2\log T}^{2\log T} h(r)dr
\end{aligned}
$$

by giving and taking back $\phi(\frac{1}{2})$ and then using symmetry $h(r)=h(-r)$.

The first summand is the Fourier transform of a L^1-function at $\log T$, it tends to 0 as $T\to\infty$. The second term tends to $\frac{1}{4}h(0)\phi(\frac{1}{2})$ as $T\to\infty$.

Finally, summating $h(t_j)$ over the discrete spectrum as well as integrating other parts against $h(r)$, we conclude the following for the truncated trace:

$$
\begin{aligned}
&\sum_{j=1}^{\infty} h(t_j)-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)\mathrm{d}r \\
&+\frac{1}{4}h(0)\phi(\frac{1}{2})+g(0)\log T+o(1).
\end{aligned}
$$

Remark. This formula is exact for large T. In our case, $\phi(\frac{1}{2})=-1$.

What about the geometric side? Only parabolic case has to be modified, hyperbolic and elliptic cases are the same as before. We will see the effect of truncation. Consider the parabolic element

$$
\gamma=\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix},\quad l\in\mathbb{Z},\quad l\neq 0.
$$

Then we have

$$
\mathrm{Tr}^T K_{\mathcal{C}}=\int_{\Gamma_\infty\backslash \mathrm{Im}z\leq T}\sum_{l\neq 0} k(z;z+l)\mathrm{d}\mu(z)+O(1). \tag{0.3}
$$

And

$$\sum_{l\neq 0}\int_0^1\int_0^T k(z;z+l)\mathrm{d}\mu(z)=\sum_{l\neq 0}\int_0^T k\Big(\big(\frac{l}{2k}\big)\Big)\frac{\mathrm{d}y}{y^2}$$
$$=\sum_{l\neq 0}|l|^{-1}\int_{(l/2T)^2}^{+\infty}k(u)\frac{\mathrm{d}u}{\sqrt{u}}=2\int_{(2T)^{-2}}^{+\infty}k(u)\sum_{1\leq l<2T\sqrt{u}}\frac{1}{l}\frac{\mathrm{d}u}{\sqrt{u}}.$$

Since

$$\sum_{1\leq l\leq x}\frac{1}{l}=\gamma+\log x+O(\frac{1}{x}),$$

where γ is the Euler constant.

We have

$$2\int_{(2T)^{-2}}^{+\infty}k(u)\Big(\log 2T\sqrt{u}+\gamma+O(\frac{1}{T\sqrt{u}})\Big)\frac{\mathrm{d}u}{\sqrt{u}}$$
$$=2\int_0^{\infty}k(u)\Big(\log 2T\sqrt{u}+\gamma\Big)\frac{\mathrm{d}u}{\sqrt{u}}+O(1)$$
$$=g(0)(\log 2T+\gamma)+\int_0^{\infty}k(u)\log u\frac{\mathrm{d}u}{\sqrt{u}},$$

where the first term is obtained by

$$\int_0^{\infty}k(u)\frac{\mathrm{d}u}{\sqrt{u}}=g(0)=q(0).$$

For the second term, we have

$$\int_0^{\infty}k(u)\log u\frac{\mathrm{d}u}{\sqrt{u}}=-\frac{1}{\pi}\int_0^{\infty}q'(v)\int_0^{v}\frac{\log u}{\sqrt{u(v-u)}}\mathrm{d}u\mathrm{d}v$$
$$=-\frac{1}{\pi}\int_0^{\infty}\int_0^1 q'(v)\frac{\log uv}{\sqrt{u(1-u)}}\mathrm{d}u\mathrm{d}v$$
$$=\frac{1}{\pi}q(0)\int_0^1\frac{\log u}{\sqrt{u(1-u)}}\mathrm{d}u-\frac{1}{\pi}\int_0^1\frac{\mathrm{d}u}{\sqrt{u(1-u)}}\int_0^{\infty}\log vq'(v)\mathrm{d}v.$$

In the last line, the first integral is $-2\pi\log 2$, the second is π, and the third is

$$\int_0^{\infty}\log vq'(v)\mathrm{d}v=\int_0^{\infty}g'(r)\log\Big(\sinh\frac{r}{2}\Big)\,\mathrm{d}r,$$

upon changing v to $(\sinh(r/2))^2$. Collecting the above together, we get

$$g(0)(\log 2T+\gamma)-\int_0^{\infty}g'(r)\log\Big(\sinh\frac{r}{2}\Big)\,\mathrm{d}r.$$

If one prefers to have an expression in terms of h, we apply the formula

$$\int_0^{\infty} g'(r)\log\left(\sinh\frac{r}{2}\right)\mathrm{d}r = g(0)(\gamma+\log 2) - \frac{1}{4}h(0) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(t)\frac{\Gamma'}{\Gamma}(s)\mathrm{d}t.$$

For the proof we write

$$g'(r) = -\frac{1}{2\pi i}\int_{\mathrm{Im}t=\varepsilon} e^{irt}h(t)t\mathrm{d}t$$

and apply

$$t\int_0^{\infty}\log\left(\sinh\frac{r}{2}\right)\mathrm{d}r = \gamma+\log 2+\frac{1}{2it}+\frac{\Gamma'}{\Gamma}(1-it).$$

Combining altogether, we obtain

$$g(0)(\gamma+\log 2)+\frac{1}{4}h(0)-\frac{1}{2\pi}\int_{-\infty}^{+\infty} h(t)\frac{\Gamma'}{\Gamma}(s)\mathrm{d}t.$$