Langlands picture of automorphic forms and L-functions
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§8 Automorphic representations for $GL_2(\mathbb{A})$ (Mar. 23)

Recall the GL_1 theory. Tate did the following.

(1) For Grossencharacter $\chi = \prod_v \chi_v$, where χ_v is unramified for almost all v, the corresponding L-function is

$$L(s, \chi) = \prod_v L(s, \chi_v).$$

(2) The local zeta function $Z(s, f_v, \chi_v)$ has analytic continuation, i.e.

$$\frac{Z(1-s, \widehat{f}_v, \chi_v^{-1})}{Z(s, f_v, \chi_v)} = \gamma(s, \chi_v, \psi_v),$$

where $\gamma(s, \chi_v, \psi_v)$ is independent the choice of f_v.

(3) The local L function satisfies

$$\gamma(s, \chi_v, \psi_v) = \varepsilon(s, \chi_v, \psi_v)\frac{L(1-s, \chi_v^{-1})}{L(s, \chi_v)},$$

and

$$L(s, \chi_v) = P(s, f_v, \chi_v)Z(s, f_v, \chi_v),$$

where $P(s, f_v, \chi_v)$ is non-zero.

(4) The global zeta function satisfies

$$Z(s, f, \chi) = Z(1-s, \widehat{f}, \chi^{-1}),$$

and the global L function has functional equation

$$1 = \varepsilon(s, \chi)\frac{L(1-s, \chi^{-1})}{L(s, \chi)}.$$

Following Tate, Jacquet-Langlands succeeded in replacing Grossencharacter $\chi = \prod_v \chi_v$ by automorphic cuspidal representation $\pi = \bigotimes_v \pi_v$. Indeed, they did this as follows.

(1) Form an automorphic representation $\pi = \bigotimes_v \pi_v$.

(2) Let W_v be a Whittaker function of $GL_2(F_v)$. The local zeta function $Z(s, W_v, g_v)$ converges in some half-plane, and

$$\frac{Z(1-s, W_v, \omega g_v)}{Z(s, W_v, g_v)}$$

has analytic continuation to all of s, where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(3) Construct the local L-function associated to π_v, and show that

$$\frac{Z(1-s, W_v, \omega g_v)}{Z(s, W_v, g_v)} = \varepsilon_v(s)\frac{L(1-s, \psi_v^{-1}\pi_v)}{L(s, \pi_v)},$$

where $\varepsilon_v(s)$ is independent of W_v. Moreover, they proved

$$\frac{Z(g, W_v, s)}{L(s, \pi_v)}$$

is almost entire. In fact, by choosing some W_v^0 for unramified v, they did show

$$L(s, \pi_v) = Z(s, W_v^0, 1).$$

(4) The global zeta function satisfies

$$Z(s, W, g) = \prod_v Z(s, W_v, g_v) = Z(1-s, W, \omega g).$$

(5) According to (4), they proved that $\pi = \bigotimes_v \pi_v$ is an automorphic cuspidal representation, i.e. an irreducible unitary representation occurring in $L_0^2(Z_{\mathbb{A}}G_F \backslash G_{\mathbb{A}}, \psi)$, if and only if its L function satisfies a simple analytic continuation and functional equation, i.e.

$$L(s, \pi) = \prod_v L(s, \pi_v) = \varepsilon(s)L(1-s, \psi^{-1} \times \pi),$$

where $\varepsilon(s) = \prod_v \varepsilon_v(s)$.

The hardest part of Jacquet-Langlands theory is to construct automorphic representation π. It is generally non-trivial to write π as tensor product of π_v over all v, where π_v is infinite dimensional representation of $GL_2(F_v) = GL_2(\mathbb{Q}_p)$. In order to do this, we need to classify all the irreducible representation of $G_{\mathbb{Q}_p} = GL_2(\mathbb{Q}_p)$.

1. Representations For Local Case

Case: Archimedean.

For $G = GL_2(\mathbb{R}) = G_\infty$ and $K = O(2, \mathbb{R})$, denote $U(\mathfrak{g})$ as the universal enveloping algebra of Lie algebra $\mathfrak{g}$ of G. Let

$$\mathscr{H}(\mathfrak{g}) = U(\mathfrak{g}) \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U(\mathfrak{g}).$$

We consider $\mathscr{H}(\mathfrak{g})$ instead of $U(\mathfrak{g})$ because $K = O(2, \mathbb{R})$ is not connected.

For V be a $\mathbb{C}$-vector space, a representation $\pi : \mathscr{H}(\mathfrak{g}) \to GL(V)$ is called "nice", if the restriction of π to Lie algebra of K_∞ has form

$$\pi\,\Big|_{\mathrm{Lie}(K_\infty)} = \bigoplus_\sigma \sigma^l,$$

where σ is finite dimensional irreducible representation of $\mathrm{Lie}(K_\infty)$ with multiplicity $l < \infty$. By describing all possible irreducible "nice" representations of $\mathscr{H}(\mathfrak{g})$, Harish-Chandra showed that representations which are "nice" and irreducible are a sub-set of the following space.

Let $H(\mu_1, \mu_2)$ be the space of functions $\varphi(g)$ on $GL_2(\mathbb{R})$ which are right K_∞-finite and satisfy

$$\varphi\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g\right) = \mu_1(t_1)\mu_2(t_2)\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}} \varphi(g).$$

Here μ_1 and μ_2 are characters (not necessarily unitary) of $\mathbb{R}^\times$, given by

$$\mu_1(t) = \mathrm{sgn}(t)^{\epsilon_1}|t|^{s_1}, \qquad \mu_2(t) = \mathrm{sgn}(t)^{\epsilon_2}|t|^{s_2}$$

where $\epsilon_1, \epsilon_2 \in \{0,1\}$ and satisfy

$$\epsilon_1 + \epsilon_2 \equiv \epsilon \bmod 2.$$

It is easy to check that $H(\mu_1,\mu_2)$ has a basis of functions $\{\varphi_n\}_{n\equiv\epsilon \bmod 2}$ given by

$$\varphi_n(g) = \mu_1(t_1)\mu_2(t_2)\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}} e^{-in\theta}, \qquad \text{for } g = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let $\pi(\mu_1,\mu_2)$ be the representation of G acting on $H(\mu_1,\mu_2)$ by right translation. We have

Theorem 8.1. *Below are the infinite dimensional irreducible nice representations of $G = GL_2(\mathbb{R})$.*

(1) *$\pi(\mu_1,\mu_2)$ is irreducible nice representation, if*

$$\mu_1\mu_2^{-1}(t) \neq |t|^p \mathrm{sgn}(t)^\epsilon, \qquad \textit{for some } p \in \mathbb{Z}.$$

(2) *If $\mu_1\mu_2^{-1}(t) = |t|^p\mathrm{sgn}(t)^\epsilon$, for some $0 < p \in \mathbb{Z}$ and $p \equiv \epsilon \bmod 2$, then $\pi(\mu_1,\mu_2)$ is not irreducible. However, it contains exactly one irreducible sub-representation generated by functions*

$$\{\cdots, \varphi_{-p-3}, \varphi_{-p-1}, \varphi_{p+1}, \varphi_{p+3}, \cdots\}. \tag{8.1}$$

(3) *Duality, if $\mu_1\mu_2^{-1}(t) = |t|^p\mathrm{sgn}(t)^\epsilon$, for some $0 > p \in \mathbb{Z}$ and $p \equiv \epsilon \bmod 2$, then $\pi(\mu_1,\mu_2)$ contains exactly one irreducible quotient representation generated by functions in (8.1).*

Case: Non-Archimedean.

For $p < \infty$, denote $G = GL_2(\mathbb{Q}_p)$. We know $K = GL_2(\mathbb{Z}_p)$ is a maximal compact open subgroup. Let $B = NA$, where

$$N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{Q}_p\right\}, \qquad A = \left\{\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, y \in \mathbb{Q}_p^\times\right\}.$$

Analogy to the case $GL_2(\mathbb{R})$, a representation π of $GL_2(\mathbb{Q}_p)$ is called "nice", if

$$\pi\big|_{\mathrm{Lie}(K)} = \bigoplus \sigma^l,$$

where σ is an irreducible representation of $\mathrm{Lie}(K)$ with multiplicity $l < \infty$. Similarly, $H(\mu_1,\mu_2)$ is defined as the space generated by locally constant functions φ which satisfy

$$\varphi\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g\right) = \mu_1(t_1)\mu_2(t_2)\left|\frac{t_1}{t_2}\right|_p^{\frac{1}{2}} \varphi(g),$$

where μ_1 and μ_2 are quasi-characters of $\mathbb{Q}_p^\times$. Denote $\pi(\mu_1,\mu_2)$ as the representation of G acting on $H(\mu_1,\mu_2)$ through right translation.

Theorem 8.2 (Jacquet-Langlands)**.** *We have*

(1) *$\pi(\mu_1,\mu_2)$ is irreducible and nice unless $\mu_1\mu_2^{-1}(t) = |t|$ or $|t|^{-1}$. In this case, it is called a* **principle series representation***.*

(2) *If $\mu_1\mu_2^{-1}(t) = |t|$, $\pi(\mu_1,\mu_2)$ contains exactly one co-dimensional one sub-representation, called a* **special representation***.*

(3) *Duality, if* $\mu_1\mu_2^{-1}(t) = |t|^{-1}$, $\pi(\mu_1, \mu_2)$ *contains exactly one quotient-representation.*

The above theorem explains that some of the irreducible nice representations are induced from the irreducible one dimensional representations of subgroup B. In fact, there exist other irreducible "nice" representations which are not accounted for by the induced representations.

Definition 8.3. Suppose (π, V) is an irreducible nice representation of G. Denote

$$V(\pi, N) = \{v \in V : \int_U \pi(n)v\mathrm{d}n = 0, \text{ for some open compact subgroup } U \text{ of } N\}.$$

π is called **supercuspidal** if $V(\pi, N) = V$.

Proposition 8.4. *Supercuspidal representations do exist. All the irreducible nice representations of G can be classified as principle, special and supercuspidal.*

Now we have the classification of all infinite dimensional irreducible "nice" representations of $GL_2(\mathbb{Q}_p)$. Recall the Grossencharacter $\psi = \prod_v \psi_v$, where ψ_v is unramified for all but finite many v. Analogy to GL_1 theory, we need to define "unramified" for the irreducible nice representation of $GL_2(\mathbb{Q}_p)$.

Definition 8.5. An irreducible nice representation π of G is called **class 1** or **spherical** if its restriction to K contains the identity representation at least one.

Theorem 8.6. *An irreducible nice representation π of G is class 1 if and only if $\pi = \pi(\mu_1, \mu_2)$ is principle, and μ_i are unramified characters of $\mathbb{Q}_p^\times$. In this case the identity representation is contained exactly once in π.*

Next, we need to consider the relation between irreducible "nice" representation and irreducible unitary representation of G. We omit details here, but give the result in the following theorem.

Theorem 8.7. *The only unitary representations which are irreducible and "nice" are*

(1) *The principle series with μ_1 and μ_2 unitary, called* **continuous series**.
(2) *The principle series with $\mu_1\mu_2^{-1}(t) = |t|^s$, where $-1 < s < 1$, called* **complementary series**.
(3) *The special representation π is unitary, if the restriction of π to the center of G is unitary.*
(4) *The supercuspidal representation π is unitary, if the restriction of π to the center of G is unitary.*

2. Representations For Global Case.

Now we define the representation π of $G_{\mathbb{A}}$ as tensor product of π_p for all $p \leq \infty$ by the following steps.

(1) For every p, we give an irreducible nice representation π_p, where π_p is class one for almost every p.
(2) For π_p of class one, we choose ξ_p^0 as a function which generates the 1-dimensional trivial subspaces of K_p.
(3) Let S_0 be the set of places containing Archimedean places and places corresponding to non-class one representations. Then $|S_0|$ is finite.

(4) For any $S \supset S_0$, set $H_S = \bigotimes_{p \in S} H_p$. For each $S' \supset S$, we have the map from H_S to $H_{S'}$ defined by

$$\xi \mapsto \xi \times (\bigotimes_{p \in S' \setminus S} \xi_p^0).$$

It implies that we have a directed system $\{H_S, S\}$ and can take the inductive limit

$$H = \lim_{\substack{\rightarrow \\ S}} H_S = \bigotimes_p H_p,$$

where if $\xi = (\xi_p) \in \bigotimes_p H_p$, $\xi_p = \xi_p^0$ for all but finite many p.

(5) Replacing H by its Hilbert completion space, we can define the unitary representation $\pi = \bigotimes_p \pi_p$ of $G_{\mathbb{A}}$ as following. For $g = (g_p) \in G_{\mathbb{A}}$, $g_p \in K_p$ for almost all p. Since $\pi = \bigotimes_p \pi_p$ where π_p is class one for almost all p. Given any $\xi = (\xi_p)$, $\xi_p = \xi_p^0$ for almost all p, we define

$$\pi(g)\xi = \bigotimes_p \pi_p(g_p)\xi_p,$$

It is easy to check that $\pi_p(g_p)\xi_p^0 = \xi_p^0$ for almost all p.