EXERCISE 3

1. Show that the invariant operators L_k (k-point-pair invariant) commute with each other.
2. By using $(\mathrm{Im} z)^s$ with $s = \frac{1}{2} + \mathrm{i}t$, show that the Selberg transform $k \longmapsto h$ is given by

$$q(v) = \int_v^\infty \frac{k(u)}{\sqrt{u-v}} \mathrm{d}u,$$

$$g(n) = \alpha_q \left(\left(\sinh \frac{n}{2} \right)^2 \right),$$

$$h(t) = \int_{-\infty}^\infty e^{\mathrm{i}nt} g(n) \mathrm{d}n.$$

Deduce that the inverse is obtained as

$$g(n) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{\mathrm{i}nt} h(t) \mathrm{d}t,$$

$$q(v) = \frac{1}{2} g \left(2\log \left(\sqrt{v+1} + \sqrt{v} \right) \right),$$

$$k(u) = -\frac{1}{\pi} \int_u^\infty \frac{q'(v)\mathrm{d}v}{\sqrt{v-u}}.$$

3. Show that the level curves $\phi =$ constant, $\varphi =$ constant of the geodesic polar coordinates are orthogonal.
4. Show that

$$\sum_{\substack{d (\bmod\ c) \\ \gcd(c,\ d)=1}} e\left(\frac{dm}{c}\right) = \sum_{\delta | \gcd(c,\ m)} \mu\left(\frac{c}{\delta}\right) \delta$$

for any integers c, m, where μ is Möbius function.

Deduce that

$$\sum_{c=1}^\infty c^{-2s} \sum_{\substack{d (\bmod c) \\ \gcd(c,d)=1}} e\left(\frac{dm}{c}\right) \frac{\sum_{d|m} d^{1-2s}}{\zeta(2s)}$$

and that

$$\zeta^*(2s)E(z;s) = \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s} + 4\sqrt{y}\sum_{n=1}^\infty \eta_{s-\frac{1}{2}}(n) K_{s-\frac{1}{2}}(2\pi n y)\cos(2\pi n x),$$

where

$$\eta_t(n) = \sum_{ab=n} \left(\frac{a}{b}\right)^t.$$