Selberg's trace formula: an introduction

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The aim of this short lecture course is to develop Selberg's trace formula for a compact hyperbolic surface $\mathcal{M}$, and discuss some of its applications. The main motivation for our studies is *quantum chaos*: the Laplace-Beltrami operator $-\Delta$ on the surface $\mathcal{M}$ represents the quantum Hamiltonian of a particle, whose classical dynamics is governed by the (strongly chaotic) geodesic flow on the unit tangent bundle of $\mathcal{M}$. The trace formula is currently the only available tool to analyze the fine structure of the spectrum of $-\Delta$; no individual formulas for its eigenvalues are known. In the case of more general quantum systems, the role of Selberg's formula is taken over by the semiclassical *Gutzwiller trace formula* [10], [7].

We begin by reviewing the trace formulas for the simplest compact manifolds, the circle $\mathbb{S}^1$ (Section 1) and the sphere $\mathbb{S}^2$ (Section 2). In both cases, the corresponding geodesic flow is integrable, and the trace formula is a consequence of the *Poisson summation formula*. In the remaining sections we shall discuss the following topics: *the Laplacian on the hyperbolic plane and isometries* (Section 3); *Green's functions* (Section 4); *Selberg's point pair invariants* (Section 5); *The ghost of the sphere* (Section 6); *Linear operators on hyperbolic surfaces* (Section 7); *A trace formula for hyperbolic cylinders and poles of the scattering matrix* (Section 8); *Back to general hyperbolic surfaces* (Section 9); *The spectrum of a compact surface, Selberg's pre-trace and trace formulas* (Section 10); *Heat kernel and Weyl's law* (Section 11); *Density of closed geodesics* (Section 12); *Trace of the resolvent* (Section 13); *Selberg's zeta function* (Section 14); *Suggestions for exercises and further reading* (Section 15).

Our main references are Hejhal's classic lecture notes [12, Chapters ONE and TWO], Balazs and Voros' excellent introduction [1], and Cartier and Voros' *nouvelle interprétation* [6]. Section 15 comprises a list of references for further reading.

These notes are based on lectures given at the International School *Quantum Chaos on Hyperbolic Manifolds*, Schloss Reisensburg (Günzburg, Germany), 4-11 October 2003.

1 Poisson summation

The Poisson summation formula reads

$$\sum_{m\in\mathbb{Z}} h(m) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} h(\rho)\, \mathrm{e}^{2\pi \imath n\rho}\, d\rho \tag{1}$$

for any sufficiently nice test function $h : \mathbb{R} \to \mathbb{C}$. One may for instance take $h \in \mathrm{C}^2(\mathbb{R})$ with $|h(\rho)| \ll (1 + |\rho|)^{-1-\delta}$ for some $\delta > 0$. (The notation $x \ll y$ means here *there exists a constant* $C > 0$ *such that* $x \leq Cy$.) Then both sums in (1) converge absolutely. (1) is proved by expanding the periodic function

$$f(\rho) = \sum_{m\in\mathbb{Z}} h(\rho + m) \tag{2}$$

in its Fourier series, and then setting $\rho = 0$.

The Poisson summation formula is our first example of a *trace formula*: The eigenvalues of the positive Laplacian $-\Delta = -\frac{d^2}{dx^2}$ on the circle $\mathbb{S}^1$ of length 2π are m^2 where $m = 0, \pm 1, \pm 2, \ldots$, with corresponding eigenfunctions $\varphi_m(x) = (2\pi)^{-1/2}\mathrm{e}^{\imath mx}$. Consider the linear operator L acting on 2π-periodic functions by

$$[Lf](x) := \int_0^{2\pi} k(x, y) f(y)\, dy \tag{3}$$

with kernel

$$k(x, y) = \sum_{m\in\mathbb{Z}} h(m)\, \varphi_m(x)\, \overline{\varphi}_m(y). \tag{4}$$

Then

$$L\varphi_m = h(m)\varphi_m \tag{5}$$

and hence the Poisson summation formula says that

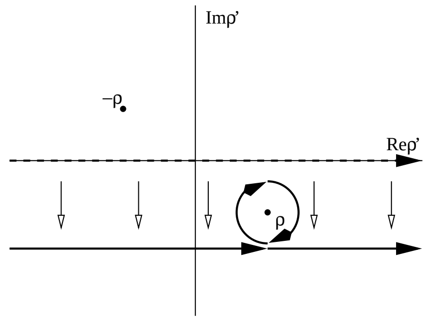
$$\operatorname{Tr} L = \sum_{m\in\mathbb{Z}} h(m) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} h(\rho)\, \mathrm{e}^{2\pi\imath n\rho}\, d\rho. \tag{6}$$

The right hand side in turn has a geometric interpretation as a sum over the periodic orbits of the geodesic flow on $\mathbb{S}^1$, whose lengths are $2\pi|n|$, $n \in \mathbb{Z}$.

An important example for a linear operator of the above type is the resolvent of the Laplacian, $(\Delta + \rho^2)^{-1}$, with $\operatorname{Im}\rho < 0$. The corresponding test function is $h(\rho') = (\rho^2 - \rho'^2)^{-1}$. Poisson summation yields in this case

$$\sum_{m\in\mathbb{Z}} (\rho^2 - m^2)^{-1} = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi\imath|n|\rho'}}{\rho^2 - \rho'^2}\, d\rho' \tag{7}$$

and by shifting the contour to $-\imath\infty$ and collecting the residue at $\rho' = \rho$,



we find

$$\sum_{m\in\mathbb{Z}} (\rho^2 - m^2)^{-1} = \frac{\pi\imath}{\rho} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-2\pi\imath|n|\rho}. \tag{8}$$

The right hand side resembles the geometric series expansion of $\cot z$ for $\operatorname{Im} z < 0$,

$$\cot z = \frac{2\imath \mathrm{e}^{-\imath z} \cos z}{1 - e^{-2\imath z}} = \imath(1 + \mathrm{e}^{-2\imath z}) \sum_{n=0}^{\infty} \mathrm{e}^{-2\imath n z} = \imath \sum_{n\in\mathbb{Z}} \mathrm{e}^{-2\imath|n|z}. \tag{9}$$

Hence

$$\sum_{m\in\mathbb{Z}} (\rho^2 - m^2)^{-1} = \frac{\pi}{\rho} \cot(\pi\rho), \tag{10}$$

which can also be written in the form

$$\frac{1}{2} \sum_{m\in\mathbb{Z}} \left[\frac{1}{\rho - m} + \frac{1}{\rho + m} \right] = \pi \cot(\pi\rho), \tag{11}$$

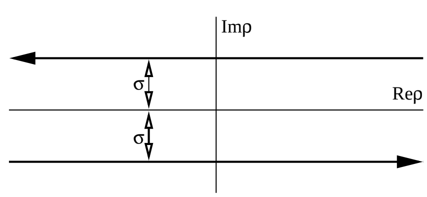
The above h is an example of a test function with particularly useful analytical properties. More generally, suppose

(i) h is analytic for $|\operatorname{Im}\rho| \leq \sigma$ for some $\sigma > 0$;
(ii) $|h(\rho)| \ll (1 + |\operatorname{Re}\rho|)^{-1-\delta}$ for some $\delta > 0$, uniformly for all ρ in the strip $|\operatorname{Im}\rho| \leq \sigma$.

Theorem 1. *If h satisfies* (i), (ii)*, then*

$$\sum_{m\in\mathbb{Z}} h(m) = \frac{1}{2\imath} \int_{\mathcal{C}_=} h(\rho) \cot(\pi\rho)\, d\rho \tag{12}$$

where the path of integration $\mathcal{C}_=$ is



Proof. The Poisson summation formula (1) may be written in the form

$$\sum_{m\in\mathbb{Z}} h(m) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} [h(\rho)+h(-\rho)]\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho. \tag{13}$$

We shift the contour of the integral to $\int_{-\infty-\imath\sigma}^{\infty-\imath\sigma}$. The geometric series expansion of $\cot z$ in (9) converges absolutely, uniformly for all z with fixed negative imaginary part. We may therefore exchange summation and integration,

$$\sum_{m\in\mathbb{Z}} h(m) = \frac{1}{2\imath}\int_{-\infty-\imath\sigma}^{\infty-\imath\sigma} [h(\rho)+h(-\rho)]\, \cot(\pi\rho)\, d\rho. \tag{14}$$

We conclude with the observation that

$$\frac{1}{2\imath}\int_{-\infty-\imath\sigma}^{\infty-\imath\sigma} h(-\rho)\, \cot(\pi\rho)\, d\rho = \frac{1}{2\imath}\int_{\infty+\imath\sigma}^{-\infty+\imath\sigma} h(\rho)\, \cot(\pi\rho)\, d\rho \tag{15}$$

since $\cot z$ is odd. □

Remark 1. This theorem can of course also be proved by shifting the lower contour in (12) across the poles to the upper contour, and evaluating the corresponding residues.

2 A trace formula for the sphere

The Laplacian on the sphere $\mathbb{S}^2$ is given by

$$\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{16}$$

where $\theta\in[0,\pi)$, $\phi\in[0,2\pi)$ are the standard spherical coordinates. The eigenvalue problem

$$(\Delta+\lambda)f = 0 \tag{17}$$

is solved by the spherical harmonics $f = Y_l^m$ for integers $l = 0,1,2,\ldots$, $m = 0,\pm1,\pm2,\ldots,\pm l$, where

$$Y_l^m(\theta,\phi) = (-1)^m\left[\frac{(2l+1)}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,\mathrm{e}^{\imath m\phi} \tag{18}$$

and P_l^m denotes the associated Legendre function of the first kind. The eigenvalue corresponding to Y_l^m is $\lambda = l(l+1)$, and hence appears with multiplicity $2l+1$. Let us label all eigenvalues (counting multiplicity) by

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty, \tag{19}$$

and set $\rho_j = \sqrt{\lambda_j + \frac{1}{4}} > 0$. For any test function $h \in \mathrm{C}^2(\mathbb{R})$ with the bound $|h(\rho)| \ll (1 + |\operatorname{Re}\rho|)^{-2-\delta}$ for some $\delta > 0$ (assume this bound also holds for the first and second derivative) we have

$$\sum_{j=0}^{\infty} h(\rho_j) = \sum_{l=0}^{\infty} (2l+1)\, h(l+\tfrac{1}{2}) \tag{20}$$

$$= \sum_{l=-\infty}^{\infty} |l+\tfrac{1}{2}|\, h(l+\tfrac{1}{2}) \tag{21}$$

$$= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} |l+\tfrac{1}{2}|\, h(l+\tfrac{1}{2}) \mathrm{e}^{2\pi \imath l n} dl \tag{22}$$

$$= 2 \sum_{n\in\mathbb{Z}} (-1)^n \int_0^\infty \rho\, h(\rho)\, \cos(2\pi n \rho)\, d\rho, \tag{23}$$

in view of the Poisson summation formula. We used the test function $|\rho| h(\rho)$ which is not continuously differentiable at $\rho = 0$. This is not a problem, since we check that (using integration by parts twice)

$$\int_0^\infty \rho\, h(\rho)\, \cos(2\pi n \rho)\, d\rho = O(n^{-2}) \tag{24}$$

hence all sums are absolutely convergent. With $\operatorname{Area}(\mathbb{S}^2) = 4\pi$, it is suggestive to write the trace formula (23) for the sphere in the form

$$\sum_{j=0}^{\infty} h(\rho_j) = \frac{\operatorname{Area}(\mathbb{S}^2)}{4\pi} \int_{\mathbb{R}} |\rho|\, h(\rho)\, d\rho + \sum_{n\neq 0} (-1)^n \int_{\mathbb{R}} |\rho|\, h(\rho)\, \mathrm{e}^{2\pi\imath n\rho}\, d\rho. \tag{25}$$

As in the trace formula for the circle, the sum on the right hand side may again be viewed as a sum over the closed geodesics of the sphere which, of course, all have lengths $2\pi|n|$. The factor $(-1)^n$ accounts for the number of conjugate points traversed by the corresponding orbit.

The sum in (23) resembles the geometric series expansion for $\tan z$ for $\operatorname{Im} z < 0$,

$$\tan z = -\cot(z + \pi/2) = -\imath \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{-2\imath|n|z}. \tag{26}$$

As remarked earlier, the sum converges uniformly for all z with fixed $\operatorname{Im} z < 0$. We have in fact the uniform bound

$$\sum_{n\in\mathbb{Z}} \left| (-1)^n \mathrm{e}^{-2\imath|n|z} \right| \leq 1 + 2 \sum_{n=1}^{\infty} \mathrm{e}^{2n \operatorname{Im} z} \leq 1 + 2 \int_0^\infty \mathrm{e}^{2x \operatorname{Im} z}\, dx = 1 - \frac{1}{\operatorname{Im} z} \tag{27}$$

which holds for all z with $\operatorname{Im} z < 0$.

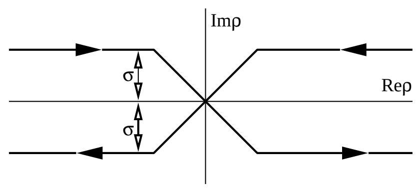
Let us use this identity to rewrite the trace formula. Assume h satisfies the following hypotheses.

(i) h is analytic for $|\operatorname{Im}\rho| \le \sigma$ for some $\sigma > 0$;
(ii) h is even, i.e., $h(-\rho) = h(\rho)$;
(iii) $|h(\rho)| \ll (1 + |\operatorname{Re}\rho|)^{-2-\delta}$ for some $\delta > 0$, uniformly for all ρ in the strip $|\operatorname{Im}\rho| \le \sigma$.

Theorem 2. *If h satisfies* (i), (ii), (iii)*, then*

$$\sum_{j=0}^{\infty} h(\rho_j) = -\frac{1}{2\imath} \int_{\mathcal{C}_\times} h(\rho)\, \rho \tan(\pi\rho)\, d\rho, \tag{28}$$

with the path of integration



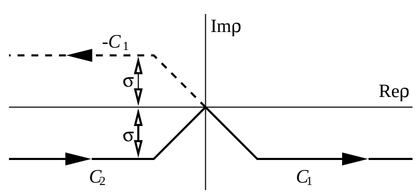
Proof. We express (23) in the form

$$\sum_{n\in\mathbb{Z}} (-1)^n \int_0^\infty \rho\, h(\rho)\, \mathrm{e}^{2\pi\imath|n|\rho}\, d\rho + \sum_{n\in\mathbb{Z}} (-1)^n \int_0^\infty \rho\, h(\rho)\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho. \tag{29}$$

which equals

$$-\sum_{n\in\mathbb{Z}} (-1)^n \int_{-\infty}^0 \rho\, h(\rho)\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho + \sum_{n\in\mathbb{Z}} (-1)^n \int_0^\infty \rho\, h(\rho)\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho. \tag{30}$$

Let us first consider the second integral. We change the path of integration to $\mathcal{C}_1$:



Due to the uniform bound (27) and

$$\int_{\mathcal{C}_1} \left|\rho\, h(\rho)[1 - (2\pi \operatorname{Im}\rho)^{-1}]\, d\rho\right| < \infty \tag{31}$$

we may exchange integration and summation, and hence the second integral in (30) evaluates to

$$\sum_{n\in\mathbb{Z}}(-1)^n \int_0^{\infty} \rho\, h(\rho)\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho = \imath \int_{\mathcal{C}_1} h(\rho)\, \rho \tan(\pi\rho)\, d\rho \tag{32}$$

The first integral in (30) is analogous, we have

$$-\sum_{n\in\mathbb{Z}}(-1)^n \int_{-\infty}^{0} \rho\, h(\rho)\, \mathrm{e}^{-2\pi\imath|n|\rho}\, d\rho = -\imath \int_{\mathcal{C}_2} h(\rho)\, \rho \tan(\pi\rho)\, d\rho \tag{33}$$

$$= \imath \int_{\mathcal{C}_2^{-1}} h(\rho)\, \rho \tan(\pi\rho)\, d\rho. \tag{34}$$

The final result is obtained by reflecting these paths at the origin, using the fact that h is even. □

Remark 2. The poles of $\tan z$ and corresponding residues can be easily worked out from (11),

$$\pi \tan(\pi\rho) = -\pi \cot\left[\pi(\rho + \tfrac{1}{2})\right] \tag{35}$$

$$= -\frac{1}{2}\sum_{l=-\infty}^{\infty}\left[\frac{1}{\rho - (l - \frac{1}{2})} + \frac{1}{\rho + (l + \frac{1}{2})}\right] \tag{36}$$

$$= -\frac{1}{2}\sum_{l=-\infty}^{\infty}\left[\frac{1}{\rho + (l + \frac{1}{2})} + \frac{1}{\rho - (l + \frac{1}{2})}\right] \tag{37}$$

(the sum has not been reordered, we have simply shifted the bracket)

$$= -\sum_{l=0}^{\infty}\left[\frac{1}{\rho + (l + \frac{1}{2})} + \frac{1}{\rho - (l + \frac{1}{2})}\right]. \tag{38}$$

Note that the extra factor ρ in the integral (28), as compared to Theorem 1, yields the multiplicity of the eigenvalues of the sphere.

3 The hyperbolic plane

In this section we briefly summarize some basic features of hyperbolic geometry; for a detailed discussion see Buser's lecture notes [5].

The hyperbolic plane $\mathbb{H}^2$ may be abstractly defined as the connected, simply connected two-dimensional Riemannian manifold with Gaussian curvature -1. Let us introduce three convenient coordinate systems for $\mathbb{H}^2$: the Poincare disk $\mathfrak{D} = \{z : |z| < 1\}$, the upper half plane $\mathfrak{H} = \{z : \operatorname{Im} z > 0\}$ and polar coordinates $(\tau, \phi) \in \mathbb{R}_{\geq 0} \times [0, 2\pi)$. In these parametrizations, the line element ds, volume element $d\mu$ and the Riemannian distance $d(z, z')$ between two points $z, z' \in \mathbb{H}^2$ are as follows.

$\mathbb{H}^2$	ds^2	$d\mu$	$\cosh d(z,z')$
$\mathfrak{D}$	$\dfrac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$	$\dfrac{4\,dx\,dy}{(1-x^2-y^2)^2}$	$1+\dfrac{2\lvert z-z'\rvert^2}{(1-\lvert z\rvert)^2(1-\lvert z'\rvert)^2}$
$\mathfrak{H}$	$\dfrac{dx^2+dy^2}{y^2}$	$\dfrac{dx\,dy}{y^2}$	$1+\dfrac{\lvert z-z'\rvert^2}{2\,\mathrm{Im}\,z\,\mathrm{Im}\,z'}$
polar	$d\tau^2+\sinh^2\tau\,d\phi^2$	$\sinh\tau\,d\tau\,d\phi$	$\cosh\tau$ [for $z=(\tau,\phi),\ z'=(0,0)$]

The *group of isometries* of $\mathbb{H}^2$, denoted by $\mathrm{Isom}(\mathbb{H}^2)$, is the group of smooth coordinate transformations which leave the Riemannian metric invariant. The group of *orientation preserving* isometries is called $\mathrm{Isom}^+(\mathbb{H}^2)$. We define the *length* of an isometry $g \in \mathrm{Isom}(\mathbb{H}^2)$ by

$$\ell_g = \ell(g) = \inf_{z\in\mathbb{H}^2} d(gz, z). \tag{39}$$

Those $g \in \mathrm{Isom}^+(\mathbb{H}^2)$ for which $\ell > 0$ are called *hyperbolic*. In the half plane model, $\mathrm{Isom}^+(\mathbb{H}^2)$ acts by fractional linear transformations,

$$g : \mathfrak{H} \to \mathfrak{H}, \qquad z \mapsto gz := \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{40}$$

(we only consider orientation-preserving isometries here). We may therefore identify g with a matrix in $\mathrm{SL}(2,\mathbb{R})$, where the matrices g and $-g$ obviously correspond to the same fractional linear transformation. $\mathrm{Isom}^+(\mathbb{H}^2)$ may thus be identified with the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. In this representation,

$$2\cosh\left(\ell_g/2\right) = \max\{|\operatorname{tr} g|, 2\}, \tag{41}$$

since every matrix $g \in \mathrm{SL}(2,\mathbb{R})$ is conjugate to one of the following three,

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{42}$$

with $b \in \mathbb{R}$, $a \in \mathbb{R}_{>0}$, $\theta \in [0, 2\pi)$.

The *Laplace-Beltrami operator* (or *Laplacian* for short) Δ of a smooth Riemannian manifold with metric

$$ds^2 = \sum_{ij} g_{jk}\, dx^j dx^k \tag{43}$$

is given by the formula

$$\Delta = \sum_{ij} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \frac{\partial}{\partial x^k}\right) \tag{44}$$

where g^{jk} are the matrix coefficients of the the inverse of the matrix (g_{jk}), and $g = |\det(g_{jk})|$. In the above coordinate systems for $\mathbb{H}^2$ the Laplacian takes the following form.

$\mathbb{H}^2$	Δ
$\mathfrak{D}$	$\frac{(1-x^2-y^2)^2}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$
$\mathfrak{H}$	$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$
polar	$\frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\left(\sinh\tau\frac{\partial}{\partial\tau}\right)+\frac{1}{\sinh^2\tau}\frac{\partial^2}{\partial\phi^2}$

One of the important properties of the Laplacian is that it commutes with every isometry $g \in \mathrm{Isom}(\mathbb{H}^2)$. That is,

$$\Delta T_g = T_g \Delta \qquad \forall g \in \mathrm{Isom}(\mathbb{H}^2). \tag{45}$$

where the left translation operator T_g acting on functions f on $\mathbb{H}^2$ is defined by

$$[T_g f](z) = f(g^{-1}z) \tag{46}$$

with $g \in \mathrm{Isom}(\mathbb{H}^2)$. Even though (45) is an intrinsic property of the Laplacian and is directly related to the invariance of the Riemannian metric under isometries, it is a useful exercise to verify (45) explicitly. To this end note that every isometry may be represented as a product of fractional linear transformations of the form $z \mapsto az$ $(a > 0)$, $z \mapsto z + b$ $(b \in \mathbb{R})$, $z \mapsto -1/z$, $z \mapsto -\overline{z}$. It is therefore sufficient to check (45) only for these four transformations.

4 Green's functions

The Green's function $G(z, w; \lambda)$ corresponding to the differential equation $(\Delta + \lambda)f(z) = 0$ is formally defined as the integral kernel of the resolvent $(\Delta + \lambda)^{-1}$, i.e., by the equation

$$(\Delta + \lambda)^{-1} f(z) = \int G(z, w; \lambda)\, f(w)\, d\mu(w) \tag{47}$$

for a suitable class of test functions f. A more precise characterization is as follows:

(G1) $G(\,\cdot\,, w; \lambda) \in \mathrm{C}^{\infty}(\mathbb{H}^2 - \{w\})$ for every fixed w;
(G2) $(\Delta + \lambda)G(z, w; \lambda) = \delta(z, w)$ for every fixed w;
(G3) as a function of (z, w), $G(z, w; \lambda)$ depends on the distance $d(z, w)$ only;
(G4) $G(z, w; \lambda) \to 0$ as $d(z, w) \to \infty$.

Here $\delta(z, w)$ is the Dirac distribution at w with respect to the measure $d\mu$. It is defined by the properties that

(D1) $\delta(z, w)d\mu(z)$ is a probability measure on $\mathbb{H}^2$;
(D2) $\int_{\mathbb{H}^2} f(z)\, \delta(z, w)\, d\mu(z) = f(w)$ for all $f \in \mathrm{C}(\mathbb{H}^2)$.

E.g., in the disk coordinates $z = x + \imath y$, $w = u + \imath v \in \mathfrak{D}$ we then have

$$\delta(z,w) = \frac{(1-x^2-y^2)^2}{4}\,\delta(x-u)\,\delta(y-v) \tag{48}$$

where $\delta(x)$ is the usual Dirac distribution with respect to Lebesgue measure on $\mathbb{R}$. In polar coordinates, where w is taken as the origin, $\tau = d(z,w)$, and

$$\delta(z,w) = \frac{\delta(\tau)}{2\pi \sinh\tau}. \tag{49}$$

Property (G2) therefore says that $(\Delta + \lambda)G(z,w;\lambda) = 0$ for $z \neq w$, and

$$\int_{d(z,w)<\epsilon} (\Delta+\lambda)G(z,w;\lambda)d\mu(z) = 1 \qquad \forall\epsilon > 0. \tag{50}$$

In view of (G3), there is a function $f \in \mathrm{C}^\infty(\mathbb{R}_{>0})$ such that $f(\tau) = G(z,w;\lambda)$. Then

$$1 = \int_{d(z,w)<\epsilon} (\Delta+\lambda)G(z,w;\lambda)d\mu(z) \tag{51}$$

$$= 2\pi \int_0^\epsilon \left(\frac{d}{d\tau}\left(\sinh\tau\frac{d}{d\tau}\right) + \lambda\sinh\tau\right) f(\tau)\,d\tau \tag{52}$$

$$= 2\pi\sinh\epsilon f'(\epsilon) + 2\pi\lambda \int_0^\epsilon \sinh\tau\, f(\tau)\,d\tau. \tag{53}$$

Taylor expansion around $\epsilon = 0$ yields $f'(\epsilon) = 1/(2\pi\epsilon) + O(1)$ and thus $f(\epsilon) = (1/2\pi)\log\epsilon + O(1)$ as $\epsilon \to 0$. Equation (G2) is therefore equivalent to

$$\begin{cases} (\Delta+\lambda)G(z,w;\lambda) = 0, & d(z,w) > 0, \\ G(z,w;\lambda) = (1/2\pi)\log d(z,w) + O(1), & d(z,w) \to 0. \end{cases} \tag{54}$$

Proposition 1. *If* $\rho \in \mathbb{C}$ *with* $\operatorname{Im}\rho < 1/2$, *and* $\lambda = \rho^2 + \frac{1}{4}$, *then*

$$G(z,w;\lambda) = -\frac{1}{2\pi} Q_{-\frac{1}{2}+\imath\rho}\big(\cosh d(z,w)\big) \tag{55}$$

satisfies (G1)-(G4), *where* Q_ν *is the Legendre function of the second kind.*

Proof. With $f(\tau) = G(z,w;\lambda)$, (54) becomes

$$\left[\frac{1}{\sinh\tau}\frac{d}{d\tau}\left(\sinh\tau\frac{d}{d\tau}\right) + \lambda\right] f(\tau) = 0. \tag{56}$$

Set $r = \cosh\tau$, $\tilde{f}(\cosh\tau) = f(\tau)$, and $\lambda = -\nu(\nu+1)$, to obtain Legendre's differential equation

$$\left[(1-r^2)\frac{d^2}{dr^2} - 2r\frac{d}{dr} + \nu(\nu+1)\right]\tilde{f}(r) = 0, \tag{57}$$

whose solutions are the associated Legendre functions $P_\nu(r)$ and $Q_\nu(r)$. Q_ν has the integral representation

$$Q_{-\frac{1}{2}+\imath\rho}(\cosh\tau) = \frac{1}{\sqrt{2}}\int_\tau^\infty \frac{\mathrm{e}^{-\imath\rho t}}{\sqrt{\cosh t - \cosh\tau}}\,dt \tag{58}$$

which converges absolutely for $\operatorname{Im}\rho < 1/2$ and $\tau > 0$. From this it is evident that $Q_{-\frac{1}{2}+\imath\rho}(\cosh\tau) \to 0$ as $\tau \to \infty$ (see also Lemma 1 below), so (G4) holds. For $t \to 0$ (ρ fixed) it has the asymptotics required in (54), cf. the well known relation

$$Q_{-\frac{1}{2}+\imath\rho}(\cosh\tau) \sim -\big(\log(\tau/2) + \gamma + \psi(\tfrac{1}{2}+\imath\rho)\big) \tag{59}$$

where γ is Euler's constant and ψ the logarithmic derivative of Euler's Γ function. $\square$

Lemma 1. *Given any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$|Q_{-\frac{1}{2}+\imath\rho}(\cosh\tau)| \le C_\epsilon |\log\tau|\,\mathrm{e}^{\tau(\operatorname{Im}\rho - \frac{1}{2}+\epsilon)} \tag{60}$$

uniformly for all $\tau > 0$ and $\rho \in \mathbb{C}$ with $\operatorname{Im}\rho < \frac{1}{2}$.

Proof. From the integral representation (58) we infer

$$|Q_{-\frac{1}{2}+\imath\rho}(\cosh\tau)| \le \frac{1}{\sqrt{2}}\int_\tau^\infty \frac{\mathrm{e}^{t\operatorname{Im}\rho}}{\sqrt{\cosh t - \cosh\tau}}\,dt \tag{61}$$

$$\le \frac{1}{\sqrt{2}}\,\mathrm{e}^{\tau(\operatorname{Im}\rho - \frac{1}{2}+\epsilon)}\int_\tau^\infty \frac{\mathrm{e}^{t(\frac{1}{2}-\epsilon)}}{\sqrt{\cosh t - \cosh\tau}}\,dt \tag{62}$$

since

$$\mathrm{e}^{t(\operatorname{Im}\rho - \frac{1}{2}+\epsilon)} \le \mathrm{e}^{\tau(\operatorname{Im}\rho - \frac{1}{2}+\epsilon)} \tag{63}$$

for all $t \ge \tau$, if $\epsilon > 0$ is chosen small enough. The remaining integral

$$\int_\tau^\infty \frac{\mathrm{e}^{t(\frac{1}{2}-\epsilon)}}{\sqrt{\cosh t - \cosh\tau}}\,dt \tag{64}$$

has a $\log\tau$ singularity at $\tau = 0$ and is otherwise uniformly bounded for all $\tau \to \infty$. $\square$

Remark 3. This is only a crude upper bound, but sufficient for our purposes.

To highlight the ρ dependence, we shall use

$$G_\rho(z,w) = -\frac{1}{2\pi}Q_{-\frac{1}{2}+\imath\rho}\big(\cosh d(z,w)\big) \tag{65}$$

instead of $G(z,w;\lambda)$.

Lemma 2. *Suppose* $f : \mathbb{H}^2 \to \mathbb{C}$ *with* $|f(z)| \leq A e^{\alpha d(z,o)}$*, with constants* $A, \alpha > 0$*. Then the integral*

$$\int_{\mathbb{H}^2} G_\rho(z,w) f(w)\, d\mu(w) \tag{66}$$

converges absolutely, uniformly in $\operatorname{Re}\rho$*, provided* $\operatorname{Im}\rho < -(\alpha + \frac{1}{2})$*. The convergence is also uniform in* z *in compact sets in* $\mathbb{H}^2$*.*

Proof. Note that $|f(w)| \leq A e^{\alpha d(o,w)} \leq A e^{\alpha d(o,z)} e^{\alpha d(z,w)}$. In polar coordinates (take w as the origin) $\tau = d(z,w)$, and $d\mu = \sinh\tau\, d\tau\, d\phi$. In view of Lemma 1, the integral (66) is bounded by

$$\ll_\epsilon \int_0^\infty |\log\tau|\; e^{-(\frac{1}{2} - \operatorname{Im}\rho - \epsilon - \alpha)\tau} \sinh\tau\, d\tau \tag{67}$$

which converges under the hypothesis on $\operatorname{Im}\rho$. □

5 Selberg's point-pair invariants

Let H be a subgroup of $\operatorname{Isom}(\mathbb{H}^2)$. An H-point-pair invariant $k : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{C}$ is defined by the relations

(K1) $k(gz, gw) = k(z,w)$ for all $g \in H$, $z, w \in \mathbb{H}^2$;
(K2) $k(w,z) = k(z,w)$ for all $z, w \in \mathbb{H}^2$.

Here we will only consider point-pair invariants which are functions of the distance between z, w, such as the Green's function $G_\rho(z,w)$ studied in the previous section. Hence $H = \operatorname{Isom}(\mathbb{H}^2)$ in this case. We sometimes use the notation $k(\tau) = k(z,w)$ with $\tau = d(z,w)$. Let us consider

$$k(z,w) = \frac{1}{\pi 1} \int_{-\infty}^{\infty} G_\rho(z,w) \rho\, h(\rho)\, d\rho \tag{68}$$

where the test function $h : \mathbb{C} \to \mathbb{C}$ satisfies the following conditions.

(H1) h is analytic for $|\operatorname{Im}\rho| \leq \sigma$ for some $\sigma > 1/2$;
(H2) h is even, i.e., $h(-\rho) = h(\rho)$;
(H3) $|h(\rho)| \ll (1 + |\operatorname{Re}\rho|)^{-2-\delta}$ for some fixed $\delta > 0$, uniformly for all ρ in the strip $|\operatorname{Im}\rho| \leq \sigma$.

For technical reasons we will sometimes use the stronger hypothesis

(H3*) $|h(\rho)| \ll_N (1 + |\operatorname{Re}\rho|)^{-N}$ for any fixed $N > 1$, uniformly for all ρ in the strip $|\operatorname{Im}\rho| \leq \sigma$.

The Fourier transform of h is

$$g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} h(\rho)\, \mathrm{e}^{-\imath\rho t} d\rho. \tag{69}$$

With the integral representation (58) one immediately finds

$$k(z,w) = -\frac{1}{\pi\sqrt{2}} \int_{\tau}^{\infty} \frac{g'(t)}{\sqrt{\cosh t - \cosh\tau}}\, dt, \quad \tau = d(z,w). \tag{70}$$

The analyticity of h and (H3) imply that g and its first derivative (*all derivatives* provided (H3*) holds) are exponentially decaying for $|t| \to \infty$. To see this, consider

$$g^{(\nu)}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} (-\imath\rho)^{\nu} h(\rho)\, \mathrm{e}^{-\imath\rho t} d\rho \tag{71}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}-\imath\sigma} (-\imath\rho)^{\nu} h(\rho)\, \mathrm{e}^{-\imath\rho t} d\rho \tag{72}$$

$$= \frac{1}{2\pi} \mathrm{e}^{-\sigma t} \int_{\mathbb{R}} [-\imath(\rho - \imath\sigma)]^{\nu} h(\rho - \imath\sigma)\, \mathrm{e}^{-\imath\rho t} d\rho. \tag{73}$$

Since, due to (H3*),

$$\int_{\mathbb{R}} |(\rho - \imath\sigma)^{\nu} h(\rho - \imath\sigma)|\, d\rho < \infty \tag{74}$$

we find

$$|g^{(\nu)}(t)| \ll_{\nu} \mathrm{e}^{-\sigma|t|}. \tag{75}$$

The point-pair invariant $k(z,w)$ gives rise to the linear operator L defined by

$$[Lf](z) := \int_{\mathbb{H}^2} k(z,w) f(w) d\mu(w). \tag{76}$$

Proposition 2. *Suppose* $f \in \mathrm{C}^2(\mathbb{H}^2)$ *is a solution of* $(\Delta + \rho^2 + \frac{1}{4})f = 0$ *with* $|\operatorname{Im}\rho| \le \sigma$ *and* $|f(z)| \le A\mathrm{e}^{\alpha d(z,o)}$*, with constants* $A > 0, 0 \le \alpha < \sigma - \frac{1}{2}$*. Then, for* h *satisfying* (H1), (H2), (H3)*,*

$$Lf = h(\rho) f. \tag{77}$$

Proof. We have

$$[Lf](z) = \int_{\mathbb{H}^2} k(z,w) f(w) d\mu(w) \tag{78}$$

$$= \frac{1}{\pi\imath} \int_{\mathbb{H}^2} \left\{ \int_{-\infty}^{\infty} G_{\rho'}(z,w)\rho'\, h(\rho')\, d\rho' \right\} f(w) d\mu(w) \tag{79}$$

$$= \frac{1}{\pi\imath} \int_{\mathbb{H}^2} \left\{ \int_{-\infty-\imath\sigma}^{\infty-\imath\sigma} G_{\rho'}(z,w)\rho'\, h(\rho')\, d\rho' \right\} f(w) d\mu(w) \tag{80}$$

where we have shifted the contour of integration by σ. Then

$$= \frac{1}{\pi \imath} \int_{-\infty - \imath\sigma}^{\infty - \imath\sigma} \left\{ \int_{\mathbb{H}^2} G_{\rho'}(z,w) f(w) d\mu(w) \right\} \rho' \, h(\rho') \, d\rho' \tag{81}$$

since the inner integral converges absolutely, uniformly in $\operatorname{Re}\rho$, see Lemma 2. We have

$$\int_{\mathbb{H}^2} G_{\rho'}(z,w) f(w) d\mu(w) = (\Delta + {\rho'}^2 + \tfrac{1}{4})^{-1} f(z) = ({\rho'}^2 - \rho^2)^{-1} f(z) \tag{82}$$

and thus

$$[Lf](z) = \frac{1}{\pi \imath} f(z) \int_{-\infty - \imath\sigma}^{\infty - \imath\sigma} \frac{\rho' h(\rho')}{{\rho'}^2 - \rho^2} \, d\rho'. \tag{83}$$

This integral converges absolutely, cf. (H3), and is easily calculated. We shift the contour from $-\imath\sigma$ to $+\imath\sigma$ and collect residues, so that

$$\frac{1}{2\pi \imath} \int_{-\infty - \imath\sigma}^{\infty - \imath\sigma} \frac{2\rho' h(\rho')}{{\rho'}^2 - \rho^2} \, d\rho' = h(\rho) + h(-\rho) + \frac{1}{2\pi \imath} \int_{-\infty + \imath\sigma}^{\infty + \imath\sigma} \frac{2\rho' h(\rho')}{{\rho'}^2 - \rho^2} \, d\rho'. \tag{84}$$

Since h is even, the integral on the right hand side equals the negative of the left hand side, and thus

$$\frac{1}{2\pi \imath} \int_{-\infty - \imath\sigma}^{\infty - \imath\sigma} \frac{2\rho' h(\rho')}{{\rho'}^2 - \rho^2} = h(\rho), \tag{85}$$

which concludes the proof. □

It is useful to define the auxiliary functions $\Phi, Q : \mathbb{R}_{\geq 0} \to \mathbb{C}$ by the relations

$$\Phi\big(2\big(\cosh\tau - 1\big)\big) = k(\tau) = k(z,w), \qquad \tau = d(z,w), \tag{86}$$

and

$$Q\big(2(\cosh t - 1)\big) = g(t). \tag{87}$$

Lemma 3. *The following statements are equivalent.*

(i) h *satisfies* (H1), (H2), (H3*).
(ii) $Q \in \mathrm{C}^\infty(\mathbb{R}_{\geq 0})$ *with*

$$\big|Q^{(\nu)}(\eta)\big| \ll_\nu \big(1+\eta\big)^{-\sigma-\nu} \qquad \forall \eta \geq 0. \tag{88}$$

Proof. Clearly $g \in \mathrm{C}^\infty(\mathbb{R})$ if and only if $Q \in \mathrm{C}^\infty(\mathbb{R}_{\geq 0})$ (this is obvious for $t \neq 0$; the problem at $t = 0$ can be resolved by expanding in Taylor series). In view of (75), the bound (88) is evident for $\nu = 0$. The νth derivative of g is of the form

$$g^{(\nu)}(t) = \sum_{j=0}^{\nu} a_{j\nu} \, \mathrm{e}^{j|t|} \, Q^{(j)}(2(\cosh t - 1)) \, (1 + O(\mathrm{e}^{-|t|})) \tag{89}$$

with suitable coefficients $a_{j\nu}$. Hence, by induction on ν,

$$\mathrm{e}^{\nu|t|}\left|Q^{(\nu)}(2(\cosh t-1))\right| \ll_\nu \left|g^{(\nu)}(t)\right| + \Big|\sum_{j=0}^{\nu-1} a_{j\nu}\,\mathrm{e}^{j|t|}\,Q^{(j)}(2(\cosh t-1))\Big| \tag{90}$$

$$\ll_\nu \left|g^{(\nu)}(t)\right| + \sum_{j=0}^{\nu-1} \mathrm{e}^{j|t|}\mathrm{e}^{(-\sigma-j)|t|} \tag{91}$$

$$\ll_\nu \mathrm{e}^{-\sigma|t|}. \tag{92}$$

This proves (i) ⇒ (ii). Conversely, (88) implies via (89) the exponential decay of g, which proves (H1). (H3*) follows from $g \in \mathrm{C}^\infty(\mathbb{R})$. □

The integral transform (70) reads in terms of the functions Q, Φ,

$$\Phi(\xi) = -\frac{1}{\pi}\int_\xi^\infty \frac{Q'(\eta)}{\sqrt{\eta-\xi}}\,d\eta. \tag{93}$$

Lemma 4. *Consider the following conditions.*

(i) $Q \in \mathrm{C}^\infty(\mathbb{R}_{\geq 0})$ *and* $\left|Q^{(\nu)}(\eta)\right| \ll_\nu \big(1+\eta\big)^{-\sigma-\nu}$.
(ii) $\Phi \in \mathrm{C}^\infty(\mathbb{R}_{\geq 0})$ *and* $\left|\Phi^{(\nu)}(\xi)\right| \ll_\nu \big(1+\xi\big)^{-\sigma-\nu-1/2+\epsilon}$.

Then (i) *implies* (ii) *for any fixed* $\epsilon > 0$, *and* (ii) *implies* (i) *for any fixed* $\epsilon < 0$.

Proof. The νth derivative of Φ is

$$\Phi^{(\nu)}(\xi) = -\frac{1}{\pi}\frac{d^\nu}{d\xi^\nu}\int_0^\infty \frac{Q'(\eta+\xi)}{\sqrt{\eta}}\,d\eta \tag{94}$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{Q^{(\nu+1)}(\eta+\xi)}{\sqrt{\eta}}\,d\eta \tag{95}$$

$$= -\frac{1}{\pi}\int_\xi^\infty \frac{Q^{(\nu+1)}(\eta)}{\sqrt{\eta-\xi}}\,d\eta. \tag{96}$$

Therefore (i) implies $\Phi \in \mathrm{C}^\infty(\mathbb{R}_{\geq 0})$. Furthermore, from (95),

$$\left|\Phi^{(\nu)}(\xi)\right| \ll_\nu \int_0^\infty \frac{\big(1+\eta+\xi\big)^{-\sigma-\nu-1}}{\sqrt{\eta}}\,d\eta \tag{97}$$

$$\ll_\nu \big(1+\xi\big)^{-\sigma-\nu-1/2+\epsilon}\int_0^\infty \frac{\big(1+\eta\big)^{-1/2-\epsilon}}{\sqrt{\eta}}\,d\eta, \tag{98}$$

for $\epsilon > 0$ small enough. The last integral converges for any $\epsilon > 0$.

The implication (ii) ⇒ (i) follows analogously from the inversion formula

$$Q(\eta)=\int_\eta^\infty \frac{\Phi(\xi)}{\sqrt{\xi-\eta}}\,d\xi. \tag{99}$$

To show that (99) is indeed consistent with (93), write (99) in the form

$$Q(\eta)=\int_{-\infty}^\infty \Phi(\eta+\xi^2)d\xi \tag{100}$$

and thus

$$Q'(\eta)=\int_{-\infty}^\infty \Phi'(\eta+\xi^2)d\xi. \tag{101}$$

The right hand side of (93) is

$$-\frac{1}{\pi}\int_\xi^\infty \frac{Q'(\eta)}{\sqrt{\eta-\xi}}\,d\eta=-\frac{1}{\pi}\int_{\mathbb{R}} Q'(\xi+\eta^2)d\eta \tag{102}$$

$$=-\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \Phi'(\xi+\eta_1^2+\eta_2^2)d\eta_1 d\eta_2, \tag{103}$$

where we have used (101) in the last step. This equals of course

$$=-2\int_0^\infty \Phi'(\xi+r^2)rdr=-\int_0^\infty \Phi'(\xi+r)dr=\Phi(\xi) \tag{104}$$

which yields the left hand side of (93). □

Proposition 3. *The following statements are equivalent.*

(i) h *satisfies* (H1), (H2), (H3*).
(ii) $k(z,w)$ *is in* $\mathrm{C}^\infty(\mathbb{H}^2\times\mathbb{H}^2)$ *with the bound on the* ν*th derivative,*

$$\left|k^{(\nu)}(\tau)\right| \ll_\nu \mathrm{e}^{-(\sigma+1/2-\epsilon)\tau} \qquad \forall\tau\geq 0, \tag{105}$$

for any fixed $\epsilon>0$.

Proof. In view of Lemmas 3 and 4, the statement (i) is equivalent to the condition for Φ, statement (ii) in Lemma 4. Since $k(z,w)=k(\tau)=\Phi\big(2(\cosh\tau-1)\big)$, the proof is exactly the same as that of Lemma 3 with $g(t)$ replaced by $k(\tau)$, and $Q\big(2(\cosh t-1)\big)$ by $\Phi\big(2(\cosh\tau-1)\big)$. □

6 The ghost of the sphere

Note that for $z=w$, the kernel $k(z,w)$ has a finite value, unlike the logarithmic divergence of the Green's function $G_\rho(z,w)$. In fact,

$$k(z,z) = -\frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{g'(t)}{\sqrt{\cosh t - 1}}\, dt \tag{106}$$
$$= -\frac{1}{2\pi} \int_0^\infty \frac{g'(t)}{\sinh(t/2)}\, dt \tag{107}$$
$$= \frac{1}{4\pi^2} \int_0^\infty \left\{ \int_{-\infty}^\infty \frac{\sin(\rho t)}{\sinh(t/2)} h(\rho)\, \rho\, d\rho \right\} dt \tag{108}$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^\infty \left\{ \int_0^\infty \frac{\sin(\rho t)}{\sinh(t/2)}\, dt \right\} h(\rho)\, \rho\, d\rho \tag{109}$$

where changing the order of integration is justified, since

$$\int_0^\infty \left| \frac{\sin(\rho t)}{\sinh(t/2)} \right| dt \le |\rho| \int_0^\infty \frac{t}{\sinh(t/2)}\, dt \ll |\rho| \tag{110}$$

and $|h(\rho)| \ll (1+|\rho|)^{-4}$, assuming (H3*). We use the geometric series expansion

$$\frac{1}{\sinh(t/2)} = \frac{2e^{-t/2}}{1-e^{-t}} = 2 \sum_{l=0}^\infty \exp\left[-\left(l + \tfrac{1}{2}\right) t \right], \tag{111}$$

so for $|\operatorname{Im} \rho| < 1/2$

$$\int_0^\infty \frac{\sin(\rho t)}{\sinh(t/2)}\, dt = \imath \sum_{l=0}^\infty \left[\frac{1}{\imath\rho - (l+\frac{1}{2})} + \frac{1}{\imath\rho + (l+\frac{1}{2})} \right] \tag{112}$$
$$= -\pi\imath \tan(\pi\imath\rho) = \pi \tanh(\pi\rho), \tag{113}$$

compare (38). We conclude

$$k(z,z) = \frac{1}{4\pi} \int_{-\infty}^\infty h(\rho) \tanh(\pi\rho)\, \rho\, d\rho. \tag{114}$$

Let us conclude this section by noting that the logarithmic divergence of the Green's function is independent of ρ, see (54). It may therefore be removed by using instead

$$G_\rho(z,w) - G_{\rho_*}(z,w) \tag{115}$$

where $\rho_* \neq \rho$ is a fixed constant in $\mathbb{C}$ with $|\operatorname{Im} \rho_*| < 1/2$. We then have from (58)

$$\lim_{w\to z} [G_\rho(z,w) - G_{\rho_*}(z,w)] = -\frac{1}{2\pi\sqrt{2}} \int_\tau^\infty \frac{e^{-\imath\rho t} - e^{-\imath\rho_* t}}{\sqrt{\cosh t - \cosh \tau}}\, dt \tag{116}$$
$$= -\frac{1}{4\pi} \int_0^\infty \frac{e^{-\imath\rho t} - e^{-\imath\rho_* t}}{\sinh(t/2)}\, dt \tag{117}$$
$$= -\frac{1}{2\pi\imath} \sum_{l=0}^\infty \left[\frac{1}{\rho - \imath(l+\frac{1}{2})} - \frac{1}{\rho_* - \imath(l+\frac{1}{2})} \right], \tag{118}$$

where we have used the geometric series expansion (111) as above. The last sum clearly converges since

$$\frac{1}{\rho - \mathrm{i}(l+\frac{1}{2})} - \frac{1}{\rho_* - \mathrm{i}(l+\frac{1}{2})} = O(l^{-2}). \tag{119}$$

Remark 4. In analogy with the trace formula for the sphere, we may view the geometric series expansion,

$$\tanh(\pi\rho) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{-2\pi|n|\rho}, \tag{120}$$

cf. (26), as a sum over closed orbits on the sphere, but now with imaginary action. These orbits have an interpretation as tunneling (or ghost) orbits.

7 Hyperbolic surfaces

Let $\mathcal{M}$ be a smooth Riemann surface (finite or infinite) of constant negative curvature which can be represented as the quotient $\Gamma\backslash\mathbb{H}^2$, where Γ is a strictly hyperbolic Fuchsian group (i.e., all elements $\gamma \in \Gamma - \{1\}$ have $\ell_\gamma > 0$). The space of square integrable functions on $\mathcal{M}$ may therefore be identified with the space of measurable functions $f : \mathbb{H}^2 \to \mathbb{C}$ satisfying the properties

$$T_\gamma f = f \quad \forall \gamma \in \Gamma \tag{121}$$

and

$$\|f\|^2 := \int_{\mathcal{F}_\Gamma} |f|^2 d\mu < \infty \tag{122}$$

where T_γ is the translation operator defined in (46) and $\mathcal{F}_\Gamma$ is any fundamental domain of Γ in $\mathbb{H}^2$. We denote this space by $\mathrm{L}^2(\Gamma\backslash\mathbb{H}^2)$. The inner product

$$\langle f_1, f_2\rangle = \int_{\mathcal{F}_\Gamma} f_1 \overline{f}_2 d\mu \tag{123}$$

makes $\mathrm{L}^2(\Gamma\backslash\mathbb{H}^2)$ a Hilbert space. Similarly, we may identify $\mathrm{C}^\infty(\Gamma\backslash\mathbb{H}^2)$ with the space of functions $f \in \mathrm{C}^\infty(\mathbb{H}^2)$ satisfying (121) (note that more care has to be taken here when Γ contains elliptic elements). Since Δ commutes with T_γ, it maps $\mathrm{C}^\infty(\Gamma\backslash\mathbb{H}^2) \to \mathrm{C}^\infty(\Gamma\backslash\mathbb{H}^2)$.

To study the spectrum of the Laplacian on $\Gamma\backslash\mathbb{H}^2$, let us consider the linear operator L of functions on $\Gamma\backslash\mathbb{H}^2$,

$$[Lf](z) := \int_{\Gamma\backslash\mathbb{H}^2} k_\Gamma(z,w) f(w) d\mu(w) \tag{124}$$

with kernel

$$k_\Gamma(z,w) = \sum_{\gamma\in\Gamma} k(\gamma z, w) \tag{125}$$

with the point-pair invariant k as defined in (68). The convergence of the sum is guaranteed by the following lemma, cf. Proposition 4 below.

Lemma 5. *For every $\delta > 0$, there is a $C_\delta > 0$ such that*

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-(1+\delta) d(\gamma z, w)} \leq C_\delta \tag{126}$$

for all $(z, w) \in \mathbb{H}^2 \times \mathbb{H}^2$.

Proof. Place a disk $\mathcal{D}_\gamma(r) = \{z' \in \mathbb{H} : d(\gamma z, z') \leq r\}$ around every point $z_\gamma = \gamma z$, and denote the area of $\mathcal{D}_\gamma(r)$ by $\mathrm{Area}(r)$. Then

$$\mathrm{e}^{-(1+\delta) d(z_\gamma, w)} \leq \frac{1}{\mathrm{Area}(r)} \int_{\mathcal{D}_\gamma(r)} \mathrm{e}^{-(1+\delta)\tilde{d}(z', w)} d\mu(z') \tag{127}$$

where

$$\tilde{d}(z', w) = \min_{z \in \mathcal{D}_\gamma(r)} d(z, w). \tag{128}$$

Because of the triangle inequality

$$\tilde{d}(z', w) \geq \min_{z \in \mathcal{D}_\gamma(r)} [d(z', w) - d(z, z')] \geq d(z', w) - 2r \tag{129}$$

for $z' \in \mathcal{D}_\gamma(r)$. Use this in (127) to obtain

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-(1+\delta) d(\gamma z, w)} \leq \frac{\mathrm{e}^{2r(1+\delta)}}{\mathrm{Area}(r)} \sum_{\gamma \in \Gamma} \int_{\mathcal{D}_\gamma(r)} \mathrm{e}^{-(1+\delta) d(z', w)} d\mu(z'). \tag{130}$$

If

$$r < \frac{1}{2} \min_{\gamma \in \Gamma - \{1\}} \ell_\gamma \tag{131}$$

the disks $\mathcal{D}_\gamma(r)$ do not overlap. (Note that $\min_{\gamma \in \Gamma - \{1\}} \ell_\gamma > 0$ since Γ is strictly hyperbolic and acts properly discontinuously on $\mathbb{H}^2$.) Therefore

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-(1+\delta) d(\gamma z, w)} \leq \frac{\mathrm{e}^{2r(1+\delta)}}{\mathrm{Area}(r)} \int_{\mathbb{H}^2} \mathrm{e}^{-(1+\delta) d(z', w)} d\mu(z') \tag{132}$$

$$= \frac{2\pi \mathrm{e}^{2r(1+\delta)}}{\mathrm{Area}(r)} \int_0^\infty \mathrm{e}^{-(1+\delta)\tau} \sinh \tau \, d\tau. \tag{133}$$

This integral converges for any $\delta > 0$. □

Proposition 4. *If h satisfies* (H1), (H2), (H3*)*, then the kernel $k_\Gamma(z, w)$ is in $\mathrm{C}^\infty(\Gamma \backslash \mathbb{H}^2 \times \Gamma \backslash \mathbb{H}^2)$, with $k_\Gamma(z, w) = k_\Gamma(w, z)$.*

Proof. Proposition 3 and Lemma 5 show that the sum over $k(\gamma z, w)$ converges absolutely and uniformly (take $\delta = \sigma - 1/2 - \epsilon > 0$). The same holds for sums over any derivative of $k(\gamma z, w)$. Hence $k_\Gamma(z, w)$ is in $\mathrm{C}^\infty(\mathbb{H}^2 \times \mathbb{H}^2)$. To prove invariance under Γ, note that

$$k_\Gamma(\gamma z, w) = \sum_{\gamma' \in \Gamma} k(\gamma'\gamma z, w) = \sum_{\gamma' \in \Gamma} k(\gamma' z, w) = k_\Gamma(z, w). \tag{134}$$

Thus $k_\Gamma(z, w)$ is a function on $\Gamma\backslash\mathbb{H}^2$ with respect to the first argument. Secondly

$$\begin{aligned} k_\Gamma(z, w) &= \sum_{\gamma' \in \Gamma} k(\gamma' z, w) = \sum_{\gamma' \in \Gamma} k(w, \gamma' z) \\ &= \sum_{\gamma' \in \Gamma} k(\gamma'^{-1} w, \gamma'^{-1}\gamma' z) = \sum_{\gamma' \in \Gamma} k(\gamma'^{-1} w, z) = k_\Gamma(w, z), \end{aligned} \tag{135}$$

which proves symmetry. Both relations imply immediately $k_\Gamma(z, \gamma w) = k_\Gamma(z, w)$. □

Proposition 5. *Suppose* $f \in \mathrm{C}^2(\Gamma\backslash\mathbb{H}^2)$ *is a solution of* $(\Delta + \rho^2 + \frac{1}{4})f = 0$ *with* $|\operatorname{Im}\rho| \leq \sigma$ *and* $|f(z)| \leq A\mathrm{e}^{\alpha d(z,o)}$, *with constants* $A > 0, 0 \leq \alpha < \sigma - \frac{1}{2}$. *Then, for* h *satisfying* (H1), (H2), (H3),

$$Lf = h(\rho)f. \tag{136}$$

Proof. Note that

$$\int_{\mathcal{F}_\Gamma} k_\Gamma(z, w) f(w) d\mu(w) = \int_{\mathbb{H}^2} k(z, w) f(w) d\mu(w) \tag{137}$$

and recall Proposition 2. □

8 A trace formula for hyperbolic cylinders

The simplest non-trivial example of a hyperbolic surface is a hyperbolic cylinder. To construct one, fix some $\gamma \in \mathrm{Isom}^+(\mathbb{H}^2)$ of length $\ell = \ell(\gamma) > 0$, and set $\Gamma = \mathfrak{Z}$, where $\mathfrak{Z}$ is the discrete subgroup generated by γ, i.e.,

$$\mathfrak{Z} = \{\gamma^n : \ n \in \mathbb{Z}\}. \tag{138}$$

We may represent $\mathfrak{Z}\backslash\mathbb{H}^2$ in halfplane coordinates, which are chosen in such a way that

$$\gamma = \begin{pmatrix} \mathrm{e}^{\ell/2} & 0 \\ 0 & \mathrm{e}^{-\ell/2} \end{pmatrix}. \tag{139}$$

A fundamental domain for the action of γ on $\mathfrak{H}$, $z \mapsto \mathrm{e}^\ell z$, is given by

$$\{z \in \mathfrak{H} : \ 1 \leq y < \mathrm{e}^\ell\}. \tag{140}$$

It is therefore evident that $\mathfrak{Z}\backslash\mathbb{H}^2$ has infinite volume. A more convenient set of parameters for the cylinder are the coordinates $(s, u) \in \mathbb{R}^2$, with

$$x = u\mathrm{e}^s, \qquad y = \mathrm{e}^s, \tag{141}$$

where the volume element reads now

$$d\mu = ds\, du. \tag{142}$$

In these coordinates, the action of γ is $(s,u) \mapsto (s+\ell, u)$, and hence a fundamental domain is

$$\mathcal{F}_{\mathfrak{Z}} = \{(s,u) \in \mathbb{R}^2 :\ 0 \le s < \ell\}. \tag{143}$$

Note that

$$\cosh d(\gamma^n z, z) = 1 + \frac{|\mathrm{e}^{n\ell} z - z|^2}{2\mathrm{e}^{n\ell} y^2} = 1 + 2\sinh^2(n\ell/2)(1+u^2) \tag{144}$$

and hence

$$k_{\mathfrak{Z}}(z,z) = k(z,z) + \sum_{n\neq 0}^{\infty} k(\gamma^n z, z) \tag{145}$$

$$= k(z,z) + 2\sum_{n=1}^{\infty} k(\gamma^n z, z) \tag{146}$$

$$= k(z,z) + 2\sum_{n=1}^{\infty} \Phi\big(4\sinh^2(n\ell/2)(1+u^2)\big). \tag{147}$$

From this we can easily work out a trace formula for the hyperbolic cylinder:

Proposition 6. *If h satisfies* (H1), (H2), (H3), *then*

$$\int_{\mathfrak{Z}\backslash\mathbb{H}^2} \big[k_{\mathfrak{Z}}(z,z) - k(z,z)\big] d\mu = \sum_{n=1}^{\infty} \frac{\ell\, g(n\ell)}{\sinh(n\ell/2)}. \tag{148}$$

Proof. We have

$$\int_{\mathbb{R}} \int_0^{\ell} \Phi\big(4\sinh^2(n\ell/2)(1+u^2)\big)\, ds\, du \tag{149}$$

$$= \frac{\ell}{2\sinh(n\ell/2)} \int_{\mathbb{R}} \Phi\big(4\sinh^2(n\ell/2) + \xi^2\big)\, d\xi \tag{150}$$

and with (100),

$$= \frac{\ell}{2\sinh(n\ell/2)}\, Q\big(4\sinh^2(n\ell/2)\big) \tag{151}$$

$$= \frac{\ell}{2\sinh(n\ell/2)}\, Q\big(2(\cosh(n\ell) - 1)\big) \tag{152}$$

which yields the right hand side of (148), cf. (87). □

Proposition 7. *If h satisfies* (H1), (H2), (H3)*, then*

$$\int_{\mathfrak{Z}\backslash\mathbb{H}^2} \big[k_{\mathfrak{Z}}(z,z) - k(z,z)\big]\, d\mu = \int_{\mathbb{R}} h(\rho)\, n_{\mathfrak{Z}}(\rho)\, d\rho \tag{153}$$

where

$$n_{\mathfrak{Z}}(\rho) = \frac{\ell}{\pi} \sum_{m=0}^{\infty} \Big\{ \exp\big[\big(m + \tfrac{1}{2} + \imath\rho\big)\,\ell\big] - 1 \Big\}^{-1} \tag{154}$$

is a meromorphic function in $\mathbb{C}$ with simple poles at the points

$$\rho_{\nu m} = \frac{\nu}{2\pi\ell} + \imath\big(m + \tfrac{1}{2}\big), \qquad \nu \in \mathbb{Z}, \quad m = 0,1,2,\ldots, \tag{155}$$

and residues $\operatorname{res}_{\rho_{\nu m}} n_{\mathfrak{Z}} = 1/(\pi\imath)$.

Proof. The geometric series expansion of $1/\sinh$ (111) yields

$$\sum_{n=1}^{\infty} \frac{\ell\, g(n\ell)}{\sinh(n\ell/2)} = \frac{\ell}{\pi} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \exp\big[-\big(m + \tfrac{1}{2} + \imath\rho\big)\, n\ell\big]\, h(\rho)\, d\rho \tag{156}$$

and using again the geometric series, this time for the sum over n,

$$\sum_{n=1}^{\infty} \exp\big[-\big(m + \tfrac{1}{2} + \imath\rho\big)\, n\ell\big] = \Big\{1 - \exp\big[-\big(m + \tfrac{1}{2} + \imath\rho\big)\,\ell\big]\Big\}^{-1} - 1 \tag{157}$$

$$= \Big\{\exp\big[\big(m + \tfrac{1}{2} + \imath\rho\big)\,\ell\big] - 1\Big\}^{-1}. \tag{158}$$

This proves the formula for $n_{\mathfrak{Z}}(\rho)$. Near each pole $\rho_{\nu m}$ we have

$$n_{\mathfrak{Z}}(\rho) \sim \frac{\ell}{\pi} \Big\{\exp\big[\big(m + \tfrac{1}{2} + \imath\rho\big)\,\ell\big] - 1\Big\}^{-1} \tag{159}$$

$$\sim \frac{1}{\pi\imath}\, \frac{1}{\rho - \big[(2\pi/\ell)\nu + \imath\big(m + \tfrac{1}{2}\big)\big]} \tag{160}$$

and so $\operatorname{res}_{\rho_{\nu m}} n_{\mathfrak{Z}} = 1/(\pi\imath)$. □

The poles of $n_{\mathfrak{Z}}(\rho)$ are called the *scattering poles* of the hyperbolic cylinder. A useful formula for $n_{\mathfrak{Z}}$ is

$$n_{\mathfrak{Z}}(\rho) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\ell\, \mathrm{e}^{-\imath\rho n\ell}}{\sinh(n\ell/2)}, \qquad \operatorname{Im}\rho < 1/2, \tag{161}$$

which follows immediately from the above proof. Furthermore, by shifting the path of integration to $-\imath\infty$, we have the identity

$$\int_{\mathbb{R}} \frac{n_{\mathfrak{Z}}(\rho')}{\rho^2 - {\rho'}^2}\, d\rho' = \frac{\pi\imath}{\rho} n_{\mathfrak{Z}}(\rho), \qquad \operatorname{Im}\rho < 0. \tag{162}$$

9 Back to general hyperbolic surfaces

Let us now show that the kernel $k_\Gamma(z,w)$ of a general hyperbolic surface $\Gamma\backslash\mathbb{H}^2$ (with Γ strictly hyperbolic) can be written as a superposition of kernels corresponding to hyperbolic cylinders.

Define the *conjugacy class* of any element $\gamma \in \Gamma$ as

$$\{\gamma\} := \{\tilde{\gamma} \in \Gamma : \tilde{\gamma} = g\gamma g^{-1} \text{ for some } g \in \Gamma\}. \tag{163}$$

Clearly the length ℓ_γ is the same for all elements in one conjugacy class. The *centralizer* of γ is

$$\mathfrak{Z}_\gamma := \{g \in \Gamma : g\gamma = \gamma g\}. \tag{164}$$

Lemma 6. *If $\gamma \in \Gamma$ is hyperbolic, then the centralizer is the infinite cyclic subgroup*

$$\mathfrak{Z}_\gamma = \{\gamma_*^n : n \in \mathbb{Z}\}, \tag{165}$$

where $\gamma_* \in \Gamma$ is uniquely determined by γ via the relation $\gamma_*^m = \gamma$ for $m \in \mathbb{N}$ as large as possible.

Proof. $\gamma \in \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Isom}^+(\mathbb{H}^2)$ is conjugate to a diagonal matrix

$$\begin{pmatrix} \mathrm{e}^{\ell_\gamma/2} & 0 \\ 0 & \mathrm{e}^{-\ell_\gamma/2} \end{pmatrix} \tag{166}$$

with $\ell_\gamma > 0$. The equation

$$\begin{pmatrix} \mathrm{e}^{\ell_\gamma/2} & 0 \\ 0 & \mathrm{e}^{-\ell_\gamma/2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\ell_\gamma/2} & 0 \\ 0 & \mathrm{e}^{-\ell_\gamma/2} \end{pmatrix} \tag{167}$$

has the only solution $b = c = 0$, $a = d^{-1}$. Hence the centralizer is a diagonal subgroup of (a conjugate of) Γ. Since Γ is discrete, the centralizer must be discrete, which forces it to be cyclic. □

Remark 5. If γ is hyperbolic and Γ strictly hyperbolic, then $\{\gamma\} \neq \{\gamma^n\}$ for all $n \neq 1$. Furthermore, the centralizers of γ and γ^n coincide.

The sum in (125) can now be expressed as

$$\sum_{\gamma\in\Gamma} k(\gamma z, w) = k(z,w) + \sum_{\gamma\in H} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} k(g^{-1}\gamma g z, w) \tag{168}$$

$$= k(z,w) + \sum_{\gamma\in H} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} k(\gamma g z, g w) \tag{169}$$

where the respective first sums run over a set H of hyperbolic elements, which contains one representative for each conjugacy class $\{\gamma\}$. We may replace this sum by a sum over *primitive* elements. If we denote by $H_* \subset H$ the subset of primitive elements, (169) equals

$$= k(z,w) + \sum_{\gamma\in H} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} k(\gamma g z, gw) \tag{170}$$

$$= k(z,w) + \sum_{\gamma\in H_*} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} \sum_{n=1}^{\infty} k(\gamma^n g z, gw) \tag{171}$$

and hence, finally,

$$k_\Gamma(z,w) - k(z,w) = \frac{1}{2} \sum_{\gamma\in H_*} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} \big\{ k_{\mathfrak{Z}_\gamma}(gz,gw) - k(gz,gw) \big\}\,; \tag{172}$$

recall that

$$k_{\mathfrak{Z}_\gamma}(z,w) = \sum_{n\in\mathbb{Z}} k(\gamma^n z, w). \tag{173}$$

10 The spectrum of a compact surface

It is well known that for any compact Riemann manifold, $-\Delta$ has positive discrete spectrum, i.e.,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty, \tag{174}$$

with corresponding eigenfunctions $\varphi_0 = \text{const}, \varphi_1, \varphi_2, \ldots \in \mathrm{C}^\infty(\Gamma\backslash\mathbb{H}^2)$, which satisfy

$$(\Delta + \lambda_j)\varphi_j = 0 \tag{175}$$

and form an orthonormal basis of $\mathrm{L}^2(\Gamma\backslash\mathbb{H}^2)$. Furthermore, since Δ is real-symmetric, the φ_j can be chosen to be real-valued. We furthermore define

$$\rho_j = \sqrt{\lambda_j - \tfrac{1}{4}}, \qquad -\pi/2 \leq \arg\rho_j < \pi/2. \tag{176}$$

If $f \in \mathrm{C}^2(\Gamma\backslash\mathbb{H}^2)$, the expansion

$$f(z) = \sum_j c_j \varphi_j(z), \qquad c_j = \langle f, \varphi_j \rangle, \tag{177}$$

converges absolutely, uniformly for all $z \in \mathbb{H}^2$. This follows from general spectral theoretic arguments, compare [12, p. 3 and Chapter THREE].

Proposition 8. *If h satisfies* (H1), (H2), (H3)*, then*

$$L\varphi_j = h(\rho_j)\varphi_j. \tag{178}$$

Proof. Apply Proposition 5. Each eigenfunction φ_j is bounded so $\alpha = 0$. Furthermore, by the positivity of $-\Delta$, we have $|\operatorname{Im}\rho| \leq 1/2 < \sigma$. □

Proposition 9. *If h satisfies* (H1), (H2), (H3*), *then*

$$k_\Gamma(z,w) = \sum_{j=0}^{\infty} h(\rho_j)\, \varphi_j(z)\, \overline{\varphi}_j(w), \tag{179}$$

which converges absolutely, uniformly in $z, w \in \mathbb{H}^2$.

Proof. The spectral expansion (177) of $k_\Gamma(z,w)$ as a function of z yields

$$k_\Gamma(z,w) = \sum_{j=0}^{\infty} c_j \varphi_j(z), \tag{180}$$

with

$$c_j = \int_{\Gamma\backslash\mathbb{H}^2} k_\Gamma(z,w)\overline{\varphi}_j(z)\, d\mu(z) = \overline{[L\varphi_j]}(w) = \overline{h}(\rho_j)\, \overline{\varphi}_j(w) = h(\rho_j)\, \overline{\varphi}_j(w). \tag{181}$$

The proof of uniform convergence follows from standard spectral theoretic arguments [12, Prop. 3.4, p.12]. □

In the case $z = w$, Proposition 9 implies immediately the following theorem.

Theorem 3 (Selberg's pre-trace formula). *If h satisfies* (H1), (H2), (H3*), *then*

$$\sum_{j=0}^{\infty} h(\rho_j)\, |\varphi_j(z)|^2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(\rho) \tanh(\pi\rho)\, \rho\, d\rho + \sum_{\gamma\in\Gamma-\{1\}} k(\gamma z, z). \tag{182}$$

which converges absolutely, uniformly in $z \in \mathbb{H}^2$.

Proof. Use (114) for the $\gamma = 1$ term. □

Using (172), the pre-trace formula (182) becomes

$$\sum_{j=0}^{\infty} h(\rho_j)\, |\varphi_j(z)|^2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(\rho) \tanh(\pi\rho)\, \rho\, d\rho \tag{183}$$

$$+ \frac{1}{2} \sum_{\gamma\in H_*} \sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma} \left\{ k_{\mathfrak{Z}_\gamma}(gz, gz) - k(gz, gz) \right\}. \tag{184}$$

Theorem 4 (Selberg's trace formula). *If h satisfies* (H1), (H2), (H3*), *then*

$$\sum_{j=0}^{\infty} h(\rho_j) = \frac{\operatorname{Area}(\mathcal{M})}{4\pi} \int_{-\infty}^{\infty} h(\rho) \tanh(\pi\rho)\, \rho\, d\rho + \sum_{\gamma\in H_*} \sum_{n=1}^{\infty} \frac{\ell_\gamma\, g(n\ell_\gamma)}{2\sinh(n\ell_\gamma/2)}, \tag{185}$$

which converges absolutely.

(We will see in the next section (Corollary 1) that the condition (H3*) may in fact be replaced by (H3).)

Proof. We integrate both sides of the pre-trace formula (182) over $\Gamma\backslash\mathbb{H}^2$. By the L^2 normalization of the eigenfunctions φ_j, the left hand side of (182) yields the left hand side of (185). The first term on the right hand side is trivial, and the second term follows from the observation that

$$\sum_{g\in\mathfrak{Z}_\gamma\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}^2} f(gz)\,d\mu = \int_{\mathfrak{Z}\backslash\mathbb{H}^2} f(z)\,d\mu, \tag{186}$$

which allows us to apply Proposition 6 to the inner sum in (184). □

Remark 6. The absolute convergence of the sum on the right hand side of (185) only requires (H1), or

$$|g(t)| \ll \mathrm{e}^{-\sigma|t|}, \qquad \forall t>0. \tag{187}$$

One way of seeing this is is that, since g is only evaluated on the discrete subset (the *length spectrum*)

$$\{\ell_\gamma : \ \gamma\in\Gamma-\{1\}\}\subset\mathbb{R}_{>0}, \tag{188}$$

we may replace g by an even $\mathrm{C}^\infty(\mathbb{R})$ function $\tilde{g}$ (for which absolute convergence is granted) so that $g(\ell_\gamma)=\tilde{g}(\ell_\gamma)$ for all γ, and

$$|\tilde{g}(t)| \ll \mathrm{e}^{-\sigma|t|}, \qquad \forall t>0. \tag{189}$$

Remark 7. We may interpret the sum over conjugacy classes in the spirit of Propositions 6 and 7: provided h satisfies (H1), (H2), (H3*), we have

$$\sum_{j=0}^{\infty} h(\rho_j) = \frac{\mathrm{Area}(\mathcal{M})}{4\pi}\int_{-\infty}^{\infty} h(\rho)\tanh(\pi\rho)\,\rho\,d\rho + \frac{1}{2}\sum_{\gamma\in H_*}\int_{\mathbb{R}} h(\rho)\,n_{\mathfrak{Z}_\gamma}(\rho)\,d\rho. \tag{190}$$

Alternatively, replace the first term on the right hand side in (185) by (108), then

$$\sum_{j=0}^{\infty} h(\rho_j) = -\frac{\mathrm{Area}(\mathcal{M})}{2\pi}\int_0^{\infty}\frac{g'(t)}{\sinh(t/2)}\,dt + \sum_{\gamma\in H_*}\sum_{n=1}^{\infty}\frac{\ell_\gamma\, g(n\ell_\gamma)}{2\sinh(n\ell_\gamma/2)}. \tag{191}$$

11 The heat kernel and Weyl's law

As a first application of the trace formula, we now prove Weyl's law for the asymptotic number of eigenvalues λ_j below a given λ,

$$N(\lambda) = \#\{j : \lambda_j \le \lambda\}, \tag{192}$$

as $\lambda\to\infty$.

Proposition 10 (Weyl's law).

$$N(\lambda) \sim \frac{\text{Area}(\mathcal{M})}{4\pi}\,\lambda, \qquad \lambda \to \infty. \tag{193}$$

Proof. For any $\beta > 0$, the test function

$$h(\rho) = \mathrm{e}^{-\beta\rho^2} \tag{194}$$

is admissible in the trace formula. The Fourier transform is

$$g(t) = \frac{\mathrm{e}^{-t^2/(2\beta)}}{\sqrt{4\pi\beta}}, \tag{195}$$

and so (185) reads in this special case (with $\lambda_j = \rho_j^2 + \frac{1}{4}$)

$$\sum_{j=0}^{\infty} \mathrm{e}^{-\beta\lambda_j} = \frac{\text{Area}(\mathcal{M})}{4\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\beta(\rho^2+\frac{1}{4})} \tanh(\pi\rho)\,\rho\,d\rho + \frac{\mathrm{e}^{-\beta/4}}{\sqrt{4\pi\beta}} \sum_{\gamma\in H_*} \sum_{n=1}^{\infty} \frac{\ell_\gamma\,\mathrm{e}^{-(n\ell_\gamma)^2/(2\beta)}}{2\sinh(n\ell_\gamma/2)}. \tag{196}$$

The sum on the right hand side clearly tends to zero in the limit $\beta \to 0$. Since $\tanh(\pi\rho) = 1 + O(\mathrm{e}^{-2\pi|\rho|})$ for all $\rho \in \mathbb{R}$, we obtain

$$\sum_{j=0}^{\infty} \mathrm{e}^{-\beta\lambda_j} = \frac{\text{Area}(\mathcal{M})}{4\pi\beta} + O(1), \qquad \beta \to 0. \tag{197}$$

The Proposition now follows from a classical Tauberian theorem [17]. □

The sum $\sum_{j=0}^{\infty} \mathrm{e}^{-\beta\lambda_j}$ represents of course the trace of the *heat kernel* $\mathrm{e}^{\beta\Delta}$.

Corollary 1. *The condition* (H3*) *in Theorem 4 and Remark 7 can be replaced by* (H3)*.*

Proof. Weyl's law implies that, for any $\delta > 0$

$$\sum_{j=0}^{\infty} (1+\lambda_j)^{-1-\delta/2} < \infty, \qquad \text{i.e.,} \qquad \sum_{j=0}^{\infty} (1+\operatorname{Re}\rho_j)^{-2-\delta} < \infty. \tag{198}$$

To prove this claim, note that

$$\sum_{j=0}^{\infty} (1+\lambda_j)^{-1-\delta/2} = \int_0^{\infty} (1+x)^{-1-\delta/2} dN(x) \tag{199}$$

$$= (1+x)^{-1-\delta/2} N(x)\Big|_{x=0}^{\infty} \tag{200}$$

$$+ \left(1+\frac{\delta}{2}\right) \int_0^{\infty} (1+x)^{-2-\delta/2} N(x)\,dx \tag{201}$$

(use integration by parts) which is finite since $N(x)$ grows linearly with x.

If h satisfies (H1), (H2), (H3), then the function $h_\epsilon(\rho) = h(\rho)\mathrm{e}^{-\epsilon\rho^2}$ clearly satisfies (H1), (H2), (H3*) for any $\epsilon > 0$, with the additional uniform bound

$$|h_\epsilon(\rho)| \ll (1 + |\operatorname{Re}\rho|)^{-2-\delta}. \tag{202}$$

where the implied constant is independent of ϵ. By repeating the calculation that leads to (75), we obtain the following estimate for the Fourier transform of h_ϵ,

$$|g_\epsilon(t)| \ll \mathrm{e}^{-\sigma|t|}, \tag{203}$$

where the implied constant is again independent of ϵ. Theorem 4 yields

$$\sum_{j=0}^{\infty} h_\epsilon(\rho_j) = \frac{\operatorname{Area}(\mathcal{M})}{4\pi} \int_{-\infty}^{\infty} h_\epsilon(\rho) \tanh(\pi\rho)\, \rho\, d\rho + \sum_{\gamma \in H_*} \sum_{n=1}^{\infty} \frac{\ell_\gamma\, g_\epsilon(n\ell_\gamma)}{2\sinh(n\ell_\gamma/2)}. \tag{204}$$

Due to the above ϵ-uniform bounds, both sides of the trace formula converge absolutely, uniformly for all $\epsilon > 0$. We may therefore take the limit $\epsilon \to 0$ inside the sums and integral.

12 The density of closed geodesics

In the previous section we have used the trace formula to obtain Weyl's law on the distribution of eigenvalues λ_j. By using the appropriate test function, one can similarly work out the asymptotic number of primitive closed geodesic with lengths $\ell_\gamma \leq L$,

$$\Pi(L) = \#\{\gamma \in H_* : \ell_\gamma \leq L\}. \tag{205}$$

In view of Remark 6 we know that, for any $\delta > 0$,

$$\sum_{\gamma \in H_*} \ell_\gamma \mathrm{e}^{-\ell_\gamma(1+\delta)} < \infty \tag{206}$$

which implies that, for any $\epsilon > 0$,

$$\Pi(L) \ll_\epsilon \mathrm{e}^{L(1+\epsilon)}. \tag{207}$$

There is in fact an a priori geometric argument (cf. [12]) which yields this bounds with $\epsilon = 0$, but the rough estimate (207) is sufficient for the following argument.

Let us consider the density of closed geodesics in the interval $[a + L, b + L]$ where a and b are fixed and $L \to \infty$. To avoid technicalities, we will here only use smoothed counting functions

$$\sum_{\gamma \in H_*} \psi_L(\ell_\gamma) = \int_0^{\infty} \psi_L(t)\, d\Pi(t) \tag{208}$$

where $\psi_L(t) = \psi(t-L)$ and $\psi \in \mathrm{C}_0^\infty(\mathbb{R})$. One may think of ψ as a smoothed characteristic function of $[a,b]$. Stronger results for true counting functions require a detailed analysis of Selberg's zeta function, which will be introduced in Section 14.

Let $\rho_0, \dots, \rho_M$ be those ρ_j with $\operatorname{Im}\rho_j < 0$. The corresponding eigenvalues $\lambda_0, \dots, \lambda_M$ are referred to as the *small* eigenvalues.

Proposition 11. *Let $\psi \in \mathrm{C}_0^\infty(\mathbb{R})$. Then, for $L > 1$,*

$$\int_0^\infty \psi_L(t)\, d\mathit{\Pi}(t) = \int_0^\infty \psi_L(t)\, d\widetilde{\mathit{\Pi}}(t) + O\left(\frac{\mathrm{e}^{L/2}}{L}\right), \tag{209}$$

where

$$d\widetilde{\mathit{\Pi}}(t) = \sum_{j=0}^{M} \frac{\mathrm{e}^{(\frac{1}{2}+\imath\rho_j)t}}{t}\, dt. \tag{210}$$

Proof. The plan is to apply the trace formula with

$$g(t) = \frac{2\sinh(t/2)}{t}\,[\psi(t-L) + \psi(-t-L)] \tag{211}$$

which is even and, for L large enough, in $\mathrm{C}_0^\infty(\mathbb{R})$. Hence its Fourier transform,

$$h(\rho) = \int_{\mathbb{R}} \frac{1}{t}\big(\mathrm{e}^{(\frac{1}{2}+\imath\rho)t} - \mathrm{e}^{(-\frac{1}{2}+\imath\rho)t}\big)\,[\psi(t-L)+\psi(-t-L)]\; dt \tag{212}$$

$$= \int_{\mathbb{R}} \frac{1}{t}\big(\mathrm{e}^{(\frac{1}{2}+\imath\rho)t} - \mathrm{e}^{(-\frac{1}{2}+\imath\rho)t} + \mathrm{e}^{(-\frac{1}{2}-\imath\rho)t} - \mathrm{e}^{(\frac{1}{2}-\imath\rho)t}\big)\psi(t-L)\, dt, \tag{213}$$

satisfies (H1), (H2), (H3). Let us begin with the integral

$$\int_{\mathbb{R}} \frac{1}{t}\,\mathrm{e}^{(\frac{1}{2}+\imath\rho)t}\psi(t-L)\, dt = \mathrm{e}^{(\frac{1}{2}+\imath\rho)L} \int_{\mathbb{R}} \frac{1}{t+L}\,\mathrm{e}^{(\frac{1}{2}+\imath\rho)t}\psi(t)\, dt. \tag{214}$$

Repeated integration by parts yields the upper bound

$$\left|\int_{\mathbb{R}} \frac{1}{t+L}\,\mathrm{e}^{(\frac{1}{2}+\imath\rho)t}\psi(t)\, dt\right| \ll_N \frac{1}{(1+|\rho|)^N} \int_{\operatorname{supp}\psi} \frac{1}{t+L}\,\mathrm{e}^{t}\, dt \tag{215}$$

$$\ll_N \frac{1}{L(1+|\rho|)^N}. \tag{216}$$

So

$$\int_{\mathbb{R}} \frac{1}{t}\,\mathrm{e}^{(\frac{1}{2}+\imath\rho)t}\psi(t-L)\, dt \ll_N \frac{\mathrm{e}^{(\frac{1}{2}-\operatorname{Im}\rho)L}}{L(1+|\rho|)^N}. \tag{217}$$

This bound is useful for $\operatorname{Im}\rho = 0$. The other corresponding integrals can be estimated in a similar way, to obtain the bounds (assume $-1/2 \le \operatorname{Im}\rho \le 0$)

$$\int_{\mathbb{R}} \frac{1}{t}\,\mathrm{e}^{(-\frac{1}{2}+\imath\rho)t}\psi(t-L)\, dt \ll_N \frac{1}{L(1+|\rho|)^N}, \tag{218}$$

$$\int_{\mathbb{R}} \frac{1}{t}\, \mathrm{e}^{(-\frac{1}{2}-\imath\rho)t}\psi(t-L)\,dt \ll_N \frac{1}{L(1+|\rho|)^N}, \tag{219}$$

and

$$\int_{\mathbb{R}} \frac{1}{t}\, \mathrm{e}^{(\frac{1}{2}-\imath\rho)t}\psi(t-L)\,dt \ll_N \frac{\mathrm{e}^{L/2}}{L(1+|\rho|)^N}. \tag{220}$$

Therefore, using the above bound with $N=3$, say, yields

$$\sum_{j=0}^{M} h(\rho_j) = \int_0^\infty \psi_L(t)\, d\widetilde{\Pi}(t) + O\left(\frac{\mathrm{e}^{L/2}}{L}\right) \tag{221}$$

and

$$\sum_{j=M+1}^{\infty} h(\rho_j) - \frac{\mathrm{Area}(\mathcal{M})}{4\pi}\int_{-\infty}^{\infty} h(\rho)\tanh(\pi\rho)\,\rho\, d\rho = O_N\left(\frac{\mathrm{e}^{L/2}}{L}\right). \tag{222}$$

The sum of the above terms equals, by the trace formula, the expression

$$\sum_{\gamma\in H_*}\sum_{n=1}^{\infty}\frac{1}{n}\psi_L(n\ell_\gamma). \tag{223}$$

The a priori bound (207) tells us that terms with $n \geq 2$ (corresponding to repetitions of primitive closed geodesics) are of lower order. To be precise,

$$\sum_{\gamma\in H_*}\sum_{n=2}^{\infty}\frac{1}{n}\psi_L(n\ell_\gamma) \ll_\epsilon \sum_{2\leq n\leq (L+b)/\ell_{\min}} \frac{1}{n}\,\mathrm{e}^{(L+b)(1+\epsilon)/n} \ll_\epsilon \mathrm{e}^{L(1+\epsilon)/2}, \tag{224}$$

where we assume that ψ_L is supported in $[a+L, b+L]$, and $\ell_{\min}$ is the length of the shortest primitive closed geodesic. Therefore

$$\sum_{\gamma\in H_*}\psi_L(\ell_\gamma) = \int_0^\infty \psi_L(t)\, d\widetilde{\Pi}(t) + O\left(\mathrm{e}^{L(1+\epsilon)/2}\right). \tag{225}$$

The leading order term as $L\to\infty$ is

$$\sum_{\gamma\in H_*}\psi_L(\ell_\gamma) \sim \int_0^\infty \psi_L(t)\,\frac{\mathrm{e}^t}{t}\,dt \ll \frac{\mathrm{e}^{L+b}}{L+b} \tag{226}$$

which leads to the improved upper bound for the sum involving repetitions,

$$\sum_{\gamma\in H_*}\sum_{n=2}^{\infty}\frac{1}{n}\psi_L(n\ell_\gamma) \ll \sum_{2\leq n\leq (L+b)/\ell_{\min}} \frac{\mathrm{e}^{(L+b)/n}}{L+b} \ll \frac{\mathrm{e}^{L/2}}{L}, \tag{227}$$

and hence leads to the desired improved error estimate in (225). □

13 Trace of the resolvent

The trace of the resolvent $R(\lambda) = (\Delta + \lambda)^{-1}$ is formally

$$\operatorname{Tr} R(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_j)^{-1} = \sum_{j=0}^{\infty} h(\rho_j), \qquad h(\rho') = (\rho^2 - {\rho'}^2)^{-1}, \tag{228}$$

where $\rho = \sqrt{\lambda - \frac{1}{4}}$ as usual. The test function h does not, however, respect condition (H3). To overcome this difficulty, we define the regularized resolvent

$$\widetilde{R}(\lambda) = (\Delta + \lambda)^{-1} - (\Delta + \lambda_*)^{-1} \tag{229}$$

for some fixed λ_*. The corresponding test function is

$$h(\rho') = (\rho^2 - {\rho'}^2)^{-1} - (\rho_*^2 - {\rho'}^2)^{-1}, \tag{230}$$

which clearly satisfies (H3), since

$$h(\rho') = \frac{\rho_*^2 - \rho^2}{(\rho^2 - {\rho'}^2)(\rho_*^2 - {\rho'}^2)} = O({\rho'}^{-4}). \tag{231}$$

We have already encountered the kernel of the regularized resolvent,

$$k(z, w) = G_\rho(z, w) - G_{\rho_*}(z, w), \tag{232}$$

in Section 6. The trace of the regularized resolvent is thus

$$\operatorname{Tr} \widetilde{R}(\rho) = \sum_{j=0}^{\infty} \left[(\rho^2 - {\rho_j}^2)^{-1} - (\rho_*^2 - {\rho_j}^2)^{-1} \right] \tag{233}$$

which, for any fixed $\rho^* \notin \{\pm\rho_j\}$, is a meromorphic function in $\mathbb{C}$ with simple poles at $\rho = \pm\rho_j$. h is analytic in the strip $|\operatorname{Im} \rho'| \leq \sigma$ provided $\sigma < |\operatorname{Im} \rho| < |\operatorname{Im} \rho^*|$ where $\sigma > 1/2$.

The trace formula (190) implies therefore (use formula (118) for the first term on the right hand side, and (162) for the second)

$$\operatorname{Tr} \widetilde{R}(\rho) = -\frac{\operatorname{Area}(\mathcal{M})}{2\pi\imath} \sum_{l=0}^{\infty} \left[\frac{1}{\rho - \imath(l + \frac{1}{2})} - \frac{1}{\rho_* - \imath(l + \frac{1}{2})} \right] + \frac{\pi\imath}{2\rho} \sum_{\gamma \in H_*} n_{\mathfrak{z}_\gamma}(\rho) + C(\rho_*). \tag{234}$$

where

$$C(\rho_*) = -\frac{\pi\imath}{2\rho_*} \sum_{\gamma \in H_*} n_{\mathfrak{z}_\gamma}(\rho_*) \tag{235}$$

converges absolutely, cf. Remark 6.

Let us rewrite this formula as

$$\frac{1}{2\rho}\sum_{\gamma\in H_*} n_{\mathfrak{z}_\gamma}(\rho) = \frac{1}{\pi\imath}\sum_{j=0}^{\infty}\left[\frac{1}{\rho^2-{\rho_j}^2}-\frac{1}{\rho_*^2-{\rho_j}^2}\right]$$
$$-\frac{\operatorname{Area}(\mathcal{M})}{2\pi^2}\sum_{l=0}^{\infty}\left[\frac{1}{\rho-\imath(l+\frac{1}{2})}-\frac{1}{\rho_*-\imath(l+\frac{1}{2})}\right]-\frac{1}{\pi\imath}C(\rho_*). \quad (236)$$

All quantities on the right hand side are meromorphic for all $\rho\in\mathbb{C}$, for every fixed $\rho_*\in\mathbb{C}$ away from the singularities (this is guaranteed for $|\operatorname{Im}\rho_*|>1/2$). Therefore (236) provides a meromorphic continuation of

$$n_\Gamma(\rho) := \sum_{\gamma\in H_*} n_{\mathfrak{z}_\gamma}(\rho) \quad (237)$$

to the whole complex plane.

Proposition 12. *The function $n_\Gamma(\rho)$ has a meromorphic continuation to the whole complex plane, with*

(i) *simple poles at $\rho=\pm\rho_j$ with residue*

$$\operatorname{res}_{\pm\rho_j} n_\Gamma = \begin{cases} 2\mu_j/(\pi\imath) & \text{if } \rho_j=0,\\ \pm\mu_j/(\pi\imath) & \text{if } \rho_j\neq 0,\end{cases} \quad (238)$$

where μ_j is the multiplicity of ρ_j.

(ii) *simple poles at $\rho=\imath(l+\frac{1}{2})$ with residue*

$$\operatorname{res}_{\imath(l+\frac{1}{2})} n_\Gamma = \frac{\operatorname{Area}(\mathcal{M})}{2\pi^2\imath}(2l+1) \quad (239)$$

(iii) *the functional relation*

$$n_\Gamma(\rho)+n_\Gamma(-\rho) = -\frac{\operatorname{Area}(\mathcal{M})}{\pi}\,\rho\tanh(\pi\rho). \quad (240)$$

Proof. (i) and (ii) are clear. (iii) follows from the identity (112). □

14 Selberg's zeta function

Selberg's zeta function is defined by

$$Z(s) = \prod_{\gamma\in H_*}\prod_{m=0}^{\infty}\left(1-\mathrm{e}^{-\ell_\gamma(s+m)}\right), \quad (241)$$

which converges absolutely for $\operatorname{Re} s>1$; this will become clear below, cf. (244) and (245). Each factor

$$\prod_{m=0}^{\infty}\left(1-\mathrm{e}^{-\ell_\gamma(s+m)}\right) \tag{242}$$

converges for all $s\in\mathbb{C}$, with zeros at

$$s=s_{\nu m}=-m+\imath(2\pi/\ell_\gamma)\nu, \qquad \nu\in\mathbb{Z}, \quad m=0,1,2,\ldots. \tag{243}$$

Note that $s_{\nu m}=\frac{1}{2}+\imath\rho_{\nu m}$ where $\rho_{\nu m}$ are the scattering poles for the hyperbolic cylinder $\mathfrak{Z}_\gamma\backslash\mathbb{H}^2$. What is more,

$$\frac{d}{ds}\log\prod_{m=0}^{\infty}\left(1-\mathrm{e}^{-\ell_\gamma(s+m)}\right)=-\ell_\gamma\sum_{m=0}^{\infty}\left(1-\mathrm{e}^{\ell_\gamma(s+m)}\right)^{-1}=\pi n_{\mathfrak{Z}_\gamma}(\rho), \tag{244}$$

with $s=\frac{1}{2}+\imath\rho$, and thus

$$\frac{Z'}{Z}(s)=\pi n_\Gamma(\rho), \qquad \operatorname{Re}s>1 \quad (\text{i.e., } \operatorname{Im}\rho<-1/2). \tag{245}$$

Recall the the genus is related to the area of $\mathcal{M}$ by $\operatorname{Area}(\mathcal{M})=4\pi(g-1)$.

Theorem 5. *The Selberg zeta function can be analytically continued to an entire function $Z(s)$ whose zeros are characterized as follows. (We divide the set of zeros into two classes,* trivial *and* non-trivial*).*

(i) *The non-trivial zeros of $Z(s)$ are located at $s=1$ and $s=\frac{1}{2}\pm\imath\rho_j$ ($j=1,2,3,\ldots$) with multiplicity*

$$\begin{cases}2\mu_j & \text{if } \rho_j=0\\ \mu_j & \text{if } \rho_j\neq 0.\end{cases} \tag{246}$$

The zero at $s=1$ (corresponding to $j=0$) has multiplicity 1.

(ii) *The trivial zeros are located at at $s=-l$, $l=0,1,2,\ldots$ and have multiplicity $2g-1$ for $l=0$ and $2(g-1)(2l+1)$ for $l>0$.*

Furthermore $Z(s)$ satisfies the functional equation

$$Z(s)=Z(1-s)\exp\left[4\pi(g-1)\int_0^{s-\frac{1}{2}}v\tan(\pi v)\,dv\right]. \tag{247}$$

Proof. Equation (236) yields

$$\frac{1}{2s-1}\frac{Z'}{Z}(s)=\sum_{j=0}^{\infty}\left[\frac{1}{(s-\frac{1}{2})^2+{\rho_j}^2}+\frac{1}{\rho_*^2-{\rho_j}^2}\right] -2(g-1)\sum_{l=0}^{\infty}\left[\frac{1}{s+l}-\frac{1}{l+\frac{1}{2}+\imath\rho_*}\right]+C(\rho_*). \tag{248}$$

Note that

$$\frac{2s-1}{(s-\frac{1}{2})^2+\rho_j{}^2}=\frac{1}{s-(\frac{1}{2}+\imath\rho_j)}+\frac{1}{s-(\frac{1}{2}-\imath\rho_j)}, \tag{249}$$

hence the corresponding residue is 1. Furthermore

$$-2(g-1)\frac{2s-1}{s+l} \tag{250}$$

has residue $2(g-1)(2l+1)$ at $s=-l$. Statements (i) and (ii) are now evident. The functional relation follows from (240), which can be written as

$$\frac{Z'}{Z}(s)+\frac{Z'}{Z}(1-s)=4\pi(g-1)(s-\tfrac{1}{2})\tan[\pi(s-\tfrac{1}{2})]. \tag{251}$$

Integrating this yields

$$\log Z(s)-\log Z(1-s)=4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv+c, \tag{252}$$

that is

$$Z(s)/Z(1-s)=\exp\left[4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv+c\right], \tag{253}$$

The constant of integration c is determined by setting $s=1/2$. Notice that the exponential is independent of the path of integration. □

One important application of the zeta function is a precise asymptotics for the number of primitive closed geodesics of length less than L, $L\to\infty$; we have [12]

$$\Pi(L)=\int_1^L d\widetilde{\Pi}(t)+O\left(\frac{\mathrm{e}^{\frac{3}{4}L}}{\sqrt{L}}\right). \tag{254}$$

The error estimate is worse than in Proposition 11, since we have replaced the smooth test functions by a characteristic function. The asymptotic relation (254) is often referred to as *Prime Geodesic Theorem*, due to its similarity with the *Prime Number Theorem*. The proof of (254) in fact follows the same strategy as in the Prime Number Theorem, where the Selberg zeta function plays the role of Riemann's zeta function.

15 Suggestions for exercises and further reading

1. **Poisson summation.**

a) The Poisson summation formula (1) reads in higher dimension d

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^d} f(\boldsymbol{m}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \widehat{f}(\boldsymbol{n}) \tag{255}$$

with the Fourier transform

$$\widehat{f}(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\rho})\, \mathrm{e}^{2\pi\imath\boldsymbol{\tau}\cdot\boldsymbol{\rho}} d\rho. \tag{256}$$

Prove (255) for a suitable class of test functions f.

b) Show that (255) can be written in the form

$$\sum_{\boldsymbol{m}\in L^*} f(\boldsymbol{m}) = \mathrm{Vol}(L\backslash\mathbb{R}^d) \sum_{\boldsymbol{n}\in L} \widehat{f}(\boldsymbol{n}) \tag{257}$$

where L is any lattice in $\mathbb{R}^d$ and L^* its dual lattice.

c) Any flat torus can be represented as the quotient $L\backslash\mathbb{R}^d$, where the Riemannian metric is the usual euclidean metric. Show that the normalized eigenfunctions of the Laplacian are

$$\varphi_{\boldsymbol{m}}(\boldsymbol{x}) = \mathrm{Vol}(L\backslash\mathbb{R}^d)^{-1/2} \mathrm{e}^{2\pi\imath\boldsymbol{m}\cdot\boldsymbol{x}} \tag{258}$$

for every $\boldsymbol{m} \in L^*$ and work out the corresponding eigenvalues λ_j.

d) Use (257) to derive a trace formula for

$$\sum_{j=0}^{\infty} h(\rho_j)$$

where $\rho_j = \sqrt{\lambda_j}$ (this formula is the famous Hardy-Voronoi formula, cf. [11]).

2. **Semiclassics.**

a) Show that for $\rho \to \infty$

$$G_\rho(z,w) = -\frac{1}{2\pi}\sqrt{\frac{\pi}{2\rho\sinh\tau}}\, \mathrm{e}^{-\imath\rho\tau-\imath\pi/4} + O(\rho^{-1}), \tag{259}$$

for all fixed $\tau = d(z,w) > 0$. Hint: divide the integral (58) into the ranges $[\tau, 2\tau)$ and $[2\tau,\infty)$. The second range is easily controlled. For the first range, use the Taylor expansion for $\cosh t$ at $t=\tau$ to expand the denominator of the integrand. Relation (259) can also be obtained from the connection of the Legendre function with the confluent hypergeometric series $F(a,b,c;z)$,

$$Q_\nu(\cosh\tau) = \sqrt{\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\frac{\mathrm{e}^{-(\nu+1)\tau}}{(1-\mathrm{e}^{-2\tau})^{1/2}} F\left(\frac{1}{2},\frac{1}{2},\nu+\frac{3}{2};\frac{1}{1-\mathrm{e}^{2\tau}}\right). \tag{260}$$

b) Show that (259) is consistent with [10, eq. (41)]. Hint: use the (s, u)-coordinates defined in (141).
c) Compare the Gutzwiller trace formula [10], [7], with the Selberg trace formula. (Analogues of the *ghost of the sphere* (Section 6) for more general systems are discussed in [2].)

3. **The Riemann-Weil explicit formula.**
   a) Compare the Selberg trace formula with the Riemann-Weil explicit formula [11, eq. (6.7)], by identifying Riemann zeros with the square-root $\sqrt{\lambda_j - \frac{1}{4}}$ of eigenvalues λ_j of the Laplacian, and logs of prime numbers with lengths of closed geodesics. See [3] for more on this.
   b) What is the analogue of the ghost of the sphere?
4. **Further reading.** In this course we have discussed Selberg's trace formula in the simplest possible set-up, for the spectrum of the Laplacian on a compact surface. The full theory, which is only outlined in Selberg's original paper [15], is developed in great detail in Hejhal's lecture notes [12], [13], where the following generalizations are discussed.
   a) The discrete subgroup Γ may contain elliptic elements, which leads to conical singularities on the surface, and reflections (i.e., orientation reversing isometries). Technically more challenging is the treatment of groups Γ which contain parabolic elements. In this case $\Gamma\backslash\mathbb{H}^2$ is no longer compact, and the spectrum has a continuous part, cf. [13].
   b) Suppose the Laplacian acts on vector valued functions $f : \mathbb{H}^2 \to \mathbb{C}^N$ which are not invariant under the action of T_γ, but satisfy
   $$T_\gamma f = \chi(\gamma) f \quad \forall \gamma \in \Gamma \tag{261}$$
   for some fixed unitary representation $\chi : \Gamma \to \mathrm{U}(N)$. The physical interpretation of this set-up, in the case $N = 1$, is that Aharonov-Bohm flux lines thread the holes of the surface.
   c) The Laplacian may act on automorphic forms of weight α, which corresponds, in physical terms, to the Hamiltonian for a constant magnetic field B perpendicular to the surface. The strength of B is proportional to α.

   I also recommend Balazs and Voros' Physics Reports article [1] and the books by Buser [4], Iwaniec [14] and Terras [16], which give a beautiful introduction to the theory. Readers interested in hyperbolic three-space will enjoy the book by Elstrodt, Grunewald and Mennicke [8]. Gelfand, Graev and Pyatetskii-Shapiro [9] take a representation-theoretic view on Selberg's trace formula.

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