Power sums of Hecke's eigenvalues of newforms and their sign changes

Jie Wu

CNRS, Université Henri Poincaré (Nancy 1)

March 11, 2009

Presented works

Presented works

[1] J. Wu, Power sums of Hecke eigenvalues and application, *Acta Arith.* (in press)

Presented works

[1] J. Wu, Power sums of Hecke eigenvalues and application, *Acta Arith.* (in press)

[2] Y.-K. Lau & J. Wu, The number of Hecke eigenvalues of same signs, *Math. Z.* (in press)

Table of contents

Table of contents

Table of contents

Table of contents

Table of contents

Table of contents

Table of contents

Definitions of modular forms, 1

- *Notation*

$$k := \text{even integer}$$

$$N := \text{squarefree integer}$$

$$\mathbb{H} := \{z \in \mathbb{C} : \Im m\, z > 0\}$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \ \text{ and } \ N \mid c \right\}$$

Definitions of modular forms, 2

- *Cusp forms*

  Let f be a holomorphic function defined in $\mathbb{H}$. We say that f is a modular form of weight k and of level N, if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad \text{and} \quad z \in \mathbb{H}.$$

Definitions of modular forms, 2

- *Cusp forms*

  Let f be a holomorphic function defined in $\mathbb{H}$. We say that f is a modular form of weight k and of level N, if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad \text{and} \quad z \in \mathbb{H}.$$

Further if $(\Im m\, z)^{k/2} f(z)$ is bounded, we say that f is a cusp form. Denote by $S_k(N)$ the set of all cusp forms of weight k and of level N.

Definitions of modular forms, 3

- *Petersson inner product*

$$\langle f, g\rangle := \int_{D_0(N)} f(z)\overline{g(z)}y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \quad (f, g \in S_k(N)),$$

where $D_0(N)$ is a fundamental domain of $\Gamma_0(N)$. $S_k(N)$ equipped with $\langle\cdot,\cdot\rangle$ is a finite dimensional Hilbert space.

Definitions of modular forms, 3

- *Petersson inner product*

$$\langle f, g\rangle := \int_{D_0(N)} f(z)\overline{g(z)}y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \quad (f, g \in S_k(N)),$$

  where $D_0(N)$ is a fundamental domain of $\Gamma_0(N)$. $S_k(N)$ equipped with $\langle\cdot,\cdot\rangle$ is a finite dimensional Hilbert space.

- *Decomposition of $S_k(N)$*

  Let $S_k^\flat(N)$ be the linear subspace of $S_k(N)$ spanned by all forms of type $f(dz)$, where $d \mid N$ and $f \in S_k(N')$ for some $N' < N$ such that $dN' \mid N$.

Definitions of modular forms, 3

- *Petersson inner product*

$$\langle f, g\rangle := \int_{D_0(N)} f(z)\overline{g(z)}y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \quad (f, g \in S_k(N)),$$

  where $D_0(N)$ is a fundamental domain of $\Gamma_0(N)$. $S_k(N)$ equipped with $\langle\cdot,\cdot\rangle$ is a finite dimensional Hilbert space.

- *Decomposition of $S_k(N)$*

  Let $S_k^\flat(N)$ be the linear subspace of $S_k(N)$ spanned by all forms of type $f(dz)$, where $d \mid N$ and $f \in S_k(N')$ for some $N' < N$ such that $dN' \mid N$.

  Let $S_k^\sharp(N)$ be the linear subspace of $S_k(N)$ orthogonal to $S_k^\flat(N)$ with respect to the Petersson inner product.

Definitions of modular forms, 4

- *Hecke's operators* $T_n : S_k(N) \longrightarrow S_k(N)$

$$f \mapsto (T_n f)(z) := \frac{1}{n} \sum_{\substack{ad=n \\ (a,N)=1}} a^k \sum_{0\leqslant b<d} f\left(\frac{az+b}{d}\right)$$

Definitions of modular forms, 4

- *Hecke's operators* $T_n : S_k(N) \to S_k(N)$

$$f \mapsto (T_n f)(z) := \frac{1}{n} \sum_{\substack{ad=n \\ (a,N)=1}} a^k \sum_{0 \leqslant b < d} f\left(\frac{az+b}{d}\right)$$

- *Newforms*

  $f \in S_k(N)$ is called *Hecke eigencuspform*, if $T_n f = c_n f$ for $(n, N) = 1$. The Hecke eigencuspforms in $S_k^\sharp(N)$ are called *newforms* (*primitive forms*). The set of all newforms, denoted by $\mathrm{H}_k^*(N)$, constitutes a base of $S_k^\sharp(N)$. We have

$$|\mathrm{H}_k^*(N)| = \frac{k-1}{12}\varphi(N) + O\big((kN)^{2/3}\big),$$

where $\varphi(N)$ is the Euler function.

Basic properties of modular forms, 1

- *Fourier development of* $f \in \mathrm{H}_k^*(N)$ *at* ∞

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

Basic properties of modular forms, 1

- *Fourier development of $f \in \mathrm{H}_k^*(N)$ at ∞*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

(a) $\lambda_f(1) = 1$,

Basic properties of modular forms, 1

- *Fourier development of* $f \in \mathrm{H}_k^*(N)$ *at* ∞

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

(a) $\lambda_f(1) = 1$,

(b) $\lambda_f(n) \in \mathbb{R}$ $(n \geqslant 1)$,

Basic properties of modular forms, 1

- *Fourier development of $f \in \mathrm{H}_k^*(N)$ at ∞*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

(a) $\lambda_f(1) = 1$,

(b) $\lambda_f(n) \in \mathbb{R}$ $(n \geqslant 1)$,

(c) $T_n f = \lambda_f(n) n^{(k-1)/2} f$ for any $n \geqslant 1$,

Basic properties of modular forms, 1

- *Fourier development of $f \in \mathrm{H}_k^*(N)$ at ∞*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

(a) $\lambda_f(1) = 1$,

(b) $\lambda_f(n) \in \mathbb{R}$ $(n \geqslant 1)$,

(c) $T_n f = \lambda_f(n) n^{(k-1)/2} f$ for any $n \geqslant 1$,

(d) for all integers $m \geqslant 1$ and $n \geqslant 1$,

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

Basic properties of modular forms, 1bis

Remark 0.

(a) For any $p \nmid N$, we have $\lambda_f(p)^2 = 1 + \lambda_f(p^2)$. Thus at least one of $\lambda_f(p)$ or $\lambda_f(p^2)$ must be bounded away from zero.

Basic properties of modular forms, 1bis

Remark 0.

(a) For any $p \nmid N$, we have $\lambda_f(p)^2 = 1 + \lambda_f(p^2)$. Thus at least one of $\lambda_f(p)$ or $\lambda_f(p^2)$ must be bounded away from zero.

(b) Let $\pi = \otimes\pi_p$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$. For $\Re s > 1$, define

$$L(s,\pi) = \prod_{p<\infty} \prod_{1\leqslant j\leqslant d} \left(1 - \alpha_\pi(p,j)p^{-s}\right)^{-1} = \sum_{n\geqslant 1} \lambda_\pi(n)n^{-s},$$

where $\alpha_\pi(p,j) \in \mathbb{C}$ such that $\alpha_\pi(p,1)\cdots\alpha_\pi(p,d) = 1$.
Yan QU (Ph. D thesis, 2008): For any prime p such that π_p is unramified,

$$|\lambda_\pi(p)| + \cdots + |\lambda_\pi(p^d)| \geqslant 1/d.$$

Basic properties of modular forms, 2

- *Deligne's inequality*

  Deligne (1974): If $f \in \mathrm{H}_k^*(N)$, then

$$|\lambda_f(n)| \leqslant d(n) \quad (n \geqslant 1)$$

  where $d(n)$ is the divisor function.

Basic properties of modular forms, 2

- *Deligne's inequality*

  Deligne (1974): If $f \in \mathrm{H}_k^*(N)$, then

$$|\lambda_f(n)| \leqslant d(n) \quad (n \geqslant 1)$$

  where $d(n)$ is the divisor function.

- *Sato-Tate's conjecture*

  For all $f \in \mathrm{H}_k^*(N)$ and $-2 \leqslant \alpha \leqslant \beta \leqslant 2$, we have

$$|\{p \leqslant x : \alpha \leqslant \lambda_f(p) \leqslant \beta\}| \sim \frac{x}{\log x} \int_\alpha^\beta \frac{\sqrt{4-t^2}}{2\pi} \mathrm{d}t.$$

  as $x \to \infty$.

Basic properties of modular forms, 3

- *Serre's result* (1981) : If $f \in \mathrm{H}_k^*(N)$, then

$$|\{p \leqslant x : \lambda_f(p) = 0\}| \ll x(\log x)^{-1-\delta} \qquad (x \geqslant 2,\ 0 < \delta < \tfrac{1}{2})$$

$$|\{n \leqslant x : \lambda_f(n) \neq 0\}| \sim \alpha x \qquad (x \to \infty,\ 0 < \alpha \overset{?}{=} 1)$$

Basic properties of modular forms, 3

- *Serre's result* (1981) : If $f \in \mathrm{H}_k^*(N)$, then

$$|\{p \leqslant x : \lambda_f(p) = 0\}| \ll x(\log x)^{-1-\delta} \qquad (x \geqslant 2,\ 0 < \delta < \tfrac{1}{2})$$

$$|\{n \leqslant x : \lambda_f(n) \neq 0\}| \sim \alpha x \qquad (x \to \infty,\ 0 < \alpha \overset{?}{=} 1)$$

- *Ramanujan's Δ-function and τ-function*

$$\Delta(z) := e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} =: \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} \in S_{12}(1)$$

Basic properties of modular forms, 3

- *Serre's result* (1981) : If $f \in \mathrm{H}_k^*(N)$, then

$$|\{p \leqslant x : \lambda_f(p) = 0\}| \ll x(\log x)^{-1-\delta} \quad (x \geqslant 2,\ 0 < \delta < \tfrac{1}{2})$$

$$|\{n \leqslant x : \lambda_f(n) \neq 0\}| \sim \alpha x \quad (x \to \infty,\ 0 < \alpha \overset{?}{=} 1)$$

- *Ramanujan's Δ-function and τ-function*

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} =: \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz} \in S_{12}(1)$$

- *Ramanujan's conjecture* (proved by Deligne, 1974)

$$|\tau(n)| \leqslant d(n) n^{11/2} \quad (n \geqslant 1)$$

Basic properties of modular forms, 3

- *Serre's result* (1981) : If $f \in \mathrm{H}_k^*(N)$, then

$$|\{p \leqslant x : \lambda_f(p) = 0\}| \ll x(\log x)^{-1-\delta} \quad (x \geqslant 2,\ 0 < \delta < \tfrac{1}{2})$$

$$|\{n \leqslant x : \lambda_f(n) \neq 0\}| \sim \alpha x \quad (x \to \infty,\ 0 < \alpha \overset{?}{=} 1)$$

- *Ramanujan's* Δ*-function and* τ*-function*

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} =: \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz} \in S_{12}(1)$$

- *Ramanujan's conjecture* (proved by Deligne, 1974)

$$|\tau(n)| \leqslant d(n) n^{11/2} \quad (n \geqslant 1)$$

- *Lehmer's conjecture* (open) : $\tau(n) \neq 0 \ (n \geqslant 1)$

Table of contents

Nonvanishing of Hecke's eigenvalues

- *Nonvanishing of* $\lambda_f(n)$

  Kowalski, Robert & Wu (2007) : For any $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$, we have

$$\sum_{\substack{x<n\leqslant x+y \\ \lambda_f(n)\neq 0}} 1 \gg_{f,\varepsilon} y$$

  for $x \geqslant x_0(f,\varepsilon)$ and $y \geqslant x^{7/17+\varepsilon}$.

Nonvanishing of Hecke's eigenvalues

- *Nonvanishing of $\lambda_f(n)$*

  Kowalski, Robert & Wu (2007) : For any $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$, we have

$$\sum_{\substack{x<n\leqslant x+y \\ \lambda_f(n)\neq 0}} 1 \gg_{f,\varepsilon} y$$

for $x \geqslant x_0(f,\varepsilon)$ and $y \geqslant x^{7/17+\varepsilon}$. In particular

$$\sum_{\substack{n\leqslant x \\ \lambda_f(n)\neq 0}} 1 \gg_f x$$

for $x \geqslant x_0(f)$.

Sign changes of Hecke's eigenvalues

- *Sign changes of* $\lambda_f(n)$

Sign changes of Hecke's eigenvalues

- *Sign changes of* $\lambda_f(n)$

Question 1 :

Which is the asymptotic comportment of

$$\mathscr{N}_f^{\pm}(x) := \sum_{\substack{n \leqslant x,\, (n,N)=1 \\ \lambda_f(n) \gtrless 0}} 1$$

as $x \to \infty$?

Moment method

- *Classical method*

  By Cauchy-Schwarz' inequality, we can write

$$\begin{aligned}\Big(\sum_{n\leqslant x,\,\lambda_f(n)\gtrless 0} |\lambda_f(n)|\Big)^2 &\leqslant \Big(\sum_{n\leqslant x,\,\lambda_f(n)\gtrless 0} 1\Big)\Big(\sum_{n\leqslant x}\lambda_f(n)^2\Big)\\ &= \mathscr{N}_f^{\pm}(x)\sum_{n\leqslant x}\lambda_f(n)^2,\end{aligned}$$

  which implies

$$\mathscr{N}_f^{\pm}(x) \geqslant \frac{\Big(\sum_{n\leqslant x,\,\lambda_f(n)\gtrless 0} |\lambda_f(n)|\Big)^2}{\sum_{n\leqslant x}\lambda_f(n)^2}.$$

First result on $\mathscr{N}_f^{\pm}(x)$

- *Kohnen, Lau & Shparlinski* (2006) :

$$\sum_{n\leqslant x,\, \lambda_f(n)\gtrless 0} |\lambda_f(n)| = \frac{1}{2}\sum_{n\leqslant x} \big(|\lambda_f(n)| \pm \lambda_f(n)\big) \gg_f \frac{x}{(\log x)^7},$$

$$\sum_{n\leqslant x} \lambda_f(n)^2 \leqslant \sum_{n\leqslant x} \tau(n)^2 \ll x(\log x)^3,$$

which imply

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \qquad \big(x \geqslant (kN)^A\big). \tag{1}$$

First result on $\mathscr{N}_f^{\pm}(x)$

- *Kohnen, Lau & Shparlinski* (2006) :

$$\sum_{n\leqslant x,\, \lambda_f(n)\gtrless 0} |\lambda_f(n)| = \frac{1}{2}\sum_{n\leqslant x} \big(|\lambda_f(n)| \pm \lambda_f(n)\big) \gg_f \frac{x}{(\log x)^7},$$

$$\sum_{n\leqslant x} \lambda_f(n)^2 \leqslant \sum_{n\leqslant x} \tau(n)^2 \ll x(\log x)^3,$$

which imply

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \qquad \big(x \geqslant (kN)^A\big). \qquad (1)$$

Remark 1. M. R. Murty (1983) :

$$\sum_{p\leqslant x,\, \lambda_f(p)\gtrless \pm 1.13} 1 \gg_f \frac{x}{\log x} \qquad (x \geqslant x_0(f)).$$

Result in short intervals

- *Kohnen, Lau & Shparlinski* (2006) :

  There are absolute constants $\vartheta < 1$ and $A > 0$ such that

$$\mathscr{N}_f^{\pm}(x + x^{\vartheta}) - \mathscr{N}_f^{\pm}(x) > 0 \tag{2}$$

  for any $f \in \mathrm{H}_k^*(N)$ and all $x \geqslant (kN)^A$.

Result in short intervals

- *Kohnen, Lau & Shparlinski* (2006) :

  There are absolute constants $\vartheta < 1$ and $A > 0$ such that

  $$\mathscr{N}_f^{\pm}(x + x^{\vartheta}) - \mathscr{N}_f^{\pm}(x) > 0 \tag{2}$$

  for any $f \in \mathrm{H}_k^*(N)$ and all $x \geqslant (kN)^A$.

  Remark 2. (a) A direct consequence of (2) is that $\lambda_f(n)$ has a sign-change in a short interval $[x, x + x^{\vartheta}]$ for $x \geqslant (kN)^A$.

Result in short intervals

- *Kohnen, Lau & Shparlinski* (2006) :

  There are absolute constants $\vartheta < 1$ and $A > 0$ such that

$$\mathscr{N}_f^{\pm}(x + x^{\vartheta}) - \mathscr{N}_f^{\pm}(x) > 0 \tag{2}$$

  for any $f \in \mathrm{H}_k^*(N)$ and all $x \geqslant (kN)^A$.

  Remark 2. (a) A direct consequence of (2) is that $\lambda_f(n)$ has a sign-change in a short interval $[x, x + x^{\vartheta}]$ for $x \geqslant (kN)^A$.
  (b) No explicit value of ϑ is evaluated. (In fact their proof gives $\vartheta = 1 - \delta$, where $\delta > 0$ is very small.)

Result in short intervals

- *Kohnen, Lau & Shparlinski* (2006) :

  There are absolute constants $\vartheta < 1$ and $A > 0$ such that

  $$\mathscr{N}_f^{\pm}(x + x^{\vartheta}) - \mathscr{N}_f^{\pm}(x) > 0 \qquad (2)$$

  for any $f \in \mathrm{H}_k^*(N)$ and all $x \geqslant (kN)^A$.

  Remark 2. (a) A direct consequence of (2) is that $\lambda_f(n)$ has a sign-change in a short interval $[x, x + x^{\vartheta}]$ for $x \geqslant (kN)^A$.
  (b) No explicit value of ϑ is evaluated. (In fact their proof gives $\vartheta = 1 - \delta$, where $\delta > 0$ is very small.)

Question 2 :

How small ϑ can be ?

Our results, 1

Our results, 1

Theorem 1. (Lau & Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f x$$

for $x \geqslant x_0(f)$.

Our results, 1

Theorem 1. (Lau & Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f x$$

for $x \geqslant x_0(f)$.

Remark 3.

(a) Theorem 1 improves (1) of Kohnen, Lau & Shparlinski, and is optimal in order of magnitude.

Our results, 1

Theorem 1. (Lau & Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f x$$

for $x \geqslant x_0(f)$.

Remark 3.

(a) Theorem 1 improves (1) of Kohnen, Lau & Shparlinski, and is optimal in order of magnitude.

(b) Our method ($\mathscr{B}$-free number method) is completely different from that of Kohnen, Lau & Shparlinski (power sum method), and very simple.

Our results, 2

Our results, 2

Theorem 2. (Lau & Wu, 2008)

There are an absolute constant $C > 0$ and a constant $x_0(k)$ such that for all $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$ we have

$$\mathscr{N}_f^{\pm}(x + CN^{1/2+\varepsilon}x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_{\varepsilon} (Nx)^{1/4-\varepsilon},$$

for $x \geqslant N^2 x_0(k)$, where the implied constant depends only on ε.

Our results, 2

Theorem 2. (Lau & Wu, 2008)

There are an absolute constant $C > 0$ and a constant $x_0(k)$ such that for all $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$ we have

$$\mathscr{N}_f^{\pm}(x + CN^{1/2+\varepsilon}x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_\varepsilon (Nx)^{1/4-\varepsilon},$$

for $x \geqslant N^2 x_0(k)$, where the implied constant depends only on ε.

Remark 4.

Our results, 2

Theorem 2. (Lau & Wu, 2008)

There are an absolute constant $C > 0$ and a constant $x_0(k)$ such that for all $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$ we have

$$\mathscr{N}_f^{\pm}(x + CN^{1/2+\varepsilon}x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_\varepsilon (Nx)^{1/4-\varepsilon},$$

for $x \geqslant N^2 x_0(k)$, where the implied constant depends only on ε.

Remark 4.

(a) Theorem 2 improves (2) of Kohnen, Lau & Sharplinski.

Our results, 2

Theorem 2. (Lau & Wu, 2008)

There are an absolute constant $C > 0$ and a constant $x_0(k)$ such that for all $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$ we have

$$\mathscr{N}_f^{\pm}(x + CN^{1/2+\varepsilon}x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_\varepsilon (Nx)^{1/4-\varepsilon},$$

for $x \geqslant N^2 x_0(k)$, where the implied constant depends only on ε.

Remark 4.

(a) Theorem 2 improves (2) of Kohnen, Lau & Sharplinski.

(b) Our method (Heath-Brown & Tsang's method) is different from that of Kohnen, Lau & Shparlinski ($\mathscr{B}$-free number method).

Our results, 2

Theorem 2. (Lau & Wu, 2008)

There are an absolute constant $C > 0$ and a constant $x_0(k)$ such that for all $f \in \mathrm{H}_k^*(N)$ and any $\varepsilon > 0$ we have

$$\mathscr{N}_f^{\pm}(x + CN^{1/2+\varepsilon}x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_{\varepsilon} (Nx)^{1/4-\varepsilon},$$

for $x \geqslant N^2 x_0(k)$, where the implied constant depends only on ε.

Remark 4.

(a) Theorem 2 improves (2) of Kohnen, Lau & Sharplinski.

(b) Our method (Heath-Brown & Tsang's method) is different from that of Kohnen, Lau & Shparlinski ($\mathscr{B}$-free number method).

(c) Wu & Zhai (2009) : 1/2 could be reduced to 3/8 "partially".

Table of contents

Rankin's power sums and his results

- *Notation*

$$S_f^*(x;r) := \sum_{n\leqslant x} |\lambda_f(n)|^{2r}.$$

- *Rankin's result* (1985)

For $f \in \mathrm{H}_k^*(N)$, $r \geqslant 0$ and $x \geqslant x_0(f,r)$, we have

$$x(\log x)^{\delta_r^{\mp}} \ll S_f^*(x;r) \ll x(\log x)^{\delta_r^{\pm}} \quad (r \in \mathcal{R}^{\mp}),$$

where

$$\mathcal{R}^- := [0,1] \cup [2,\infty), \qquad \delta_r^- := 2^{r-1} - 1,$$
$$\mathcal{R}^+ := [1,2], \qquad \delta_r^+ := 2^r(2^r + 3^{2-r})/10 - 1.$$

Tenenbaum's result

- *Tenenbaum's result* (2007)

Let τ be Ramanujan's function, $\lambda_\tau(n) = \tau(n)n^{-11/2}$. Then

$$S_\tau^*(x; \tfrac{1}{2}) = \sum_{n \leqslant x} |\lambda_\tau(n)| \ll x(\log x)^{\rho_{1/2}^+} \qquad (3)$$

where

$$\rho_{1/2}^+ := \tfrac{102+7\sqrt{21}}{210}\big(\tfrac{6-\sqrt{21}}{5}\big)^{1/2} + \tfrac{102-7\sqrt{21}}{210}\big(\tfrac{6+\sqrt{21}}{5}\big)^{1/2} - \tfrac{33}{35}.$$

Remark 5.

(a) Tenenbaum's $\rho_{1/2}^+ \approx -0.11$ improves Rankin's $\delta_{1/2}^+ \approx -0.06$.

(b) His method can be easily generalized for obtaining upper bound of $S_f^*(x; r)$ for all $f \in \mathrm{H}_k^*(N)$ and $r > 0$ and does not give lower bound of $S_f^*(x; r)$.

Our results

Theorem 3. (Wu, 2008)

For any $f \in \mathrm{H}_k^*(N)$, we have

$$x(\log x)^{\rho_r^{\mp}} \ll_{f,r} S_f^*(x;r) \ll_{f,r} x(\log x)^{\rho_r^{\pm}} \quad (r \in \mathscr{R}^{\mp})$$

for $x \geqslant x_0(f,r)$, where

$$\mathscr{R}^- := [0,1] \cup [2,3] \cup [4,\infty), \qquad \mathscr{R}^+ := [1,2] \cup [3,4],$$

and

$$\rho_r^- := \frac{3^{r-1}-1}{2},$$
$$\rho_r^+ := \frac{102+7\sqrt{21}}{210}\Big(\frac{6-\sqrt{21}}{5}\Big)^r + \frac{102-7\sqrt{21}}{210}\Big(\frac{6+\sqrt{21}}{5}\Big)^r + \frac{4^r}{35} - 1.$$

Remarks

(a) Iwaniec asked the question about what best can one do on

$$S_f^*(x; \tfrac{1}{2}) := \sum_{n \leqslant x} |\lambda_f(n)| \ ?$$

(AIM, workshop on GL_3, problem session, November 2008, USA)

Remarks

(a) Iwaniec asked the question about what best can one do on

$$S_f^*(x;\tfrac{1}{2}) := \sum_{n\leqslant x} |\lambda_f(n)| \ ?$$

(AIM, workshop on GL_3, problem session, November 2008, USA)

(b) Rankin (1985) : Assuming Sato-Tate' conjecture, then

$$S_f^*(x;r) \sim C_r(f)x(\log x)^{\theta_r} \quad (x\to\infty),$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r\Gamma(r+1/2)}{\sqrt{\pi}\Gamma(r+2)} - 1.$$

Remarks

(a) Iwaniec asked the question about what best can one do on

$$S_f^*(x; \tfrac{1}{2}) := \sum_{n \leqslant x} |\lambda_f(n)| \ ?$$

(AIM, workshop on GL_3, problem session, November 2008, USA)

(b) Rankin (1985) : Assuming Sato-Tate' conjecture, then

$$S_f^*(x; r) \sim C_r(f) x (\log x)^{\theta_r} \quad (x \to \infty),$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r \Gamma(r + 1/2)}{\sqrt{\pi}\,\Gamma(r+2)} - 1.$$

Comparison : $\delta_{1/2}^- \approx -0.292,\ \rho_{1/2}^- \approx -0.211,\ \theta_{1/2} \approx -0.151.$

Remarks

(a) Iwaniec asked the question about what best can one do on

$$S_f^*(x;\tfrac{1}{2}) := \sum_{n\leqslant x} |\lambda_f(n)| \ ?$$

(AIM, workshop on GL_3, problem session, November 2008, USA)

(b) Rankin (1985) : Assuming Sato-Tate' conjecture, then

$$S_f^*(x;r) \sim C_r(f)x(\log x)^{\theta_r} \quad (x\to\infty),$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r\Gamma(r+1/2)}{\sqrt{\pi}\Gamma(r+2)} - 1.$$

Comparison : $\delta_{1/2}^- \approx -0.292, \ \rho_{1/2}^- \approx -0.211, \ \theta_{1/2} \approx -0.151.$

Application of Theorem 3

Corollary 1. (Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$ and $x \geqslant x_0(f)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{1-1/\sqrt{3}}}.$$

Application of Theorem 3

Corollary 1. (Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$ and $x \geqslant x_0(f)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{1-1/\sqrt{3}}}.$$

Remark 6.

(a) Corollary 1 is non trivial with respect to Murty's result above.

Application of Theorem 3

Corollary 1. (Wu, 2008)

For all $f \in \mathrm{H}_k^*(N)$ and $x \geqslant x_0(f)$, we have

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{1-1/\sqrt{3}}}.$$

Remark 6.

(a) Corollary 1 is non trivial with respect to Murty's result above.

(b) Assuming Sato-Tate's conjecture, the exponent $1 - 1/\sqrt{3} \approx 0.422$ could be improved to $2 - 16/(3\pi) \approx 0.302$.

Table of contents

Mean-value theorem

Theorem A. (Tenenbaum, 2007)

Let $g : \mathbb{N} \to \mathbb{R}^+$ be a multiplicative function such that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} g(n)^2 \ \text{(exists)}, \quad \sum_p \frac{g(p)^4}{p^2} < \infty, \quad \sum_{p\leqslant x} g(p) \log p \ll x.$$

Then for $x \geqslant 1$ we have

$$\sum_{n\leqslant x} g(n) \ll x \exp\left\{ - \sum_{p\leqslant x} \frac{(g(p)-1)^2}{2p} \right\}.$$

Begining of the proof of (3)

Theorem A is applicable to $g(n) = |\lambda_\tau(n)| = |\tau(n)|n^{-11/2}$, since

$$\sum_{n\leqslant x} \lambda_\tau(n)^2 = C_f x + O(x^{3/5}) \quad \text{(Rankin-Selberg)},$$

$$|\lambda_\tau(n)| \leqslant d(n) \quad \text{(Deligne)}.$$

Thus for $x \geqslant 1$,

$$\begin{aligned}\sum_{n\leqslant x} |\lambda_\tau(n)| &\ll x \exp\left\{ -\sum_{p\leqslant x} \frac{(|\lambda_\tau(p)|-1)^2}{2p}\right\} \\ &\ll x \exp\left\{ -\log_2 x + \frac{1}{2}\sum_{p\leqslant x} \frac{|\lambda_\tau(p)|}{p}\right\}.\end{aligned}$$

A optimilisation problem, 1

Question 3 :

Suppose that as $x \to \infty$,

$$\sum_{p \leqslant x} \frac{\lambda_\tau(p)^{2j}}{p} = m_j \log_2 x + O(1) \qquad (1 \leqslant j \leqslant 4),$$

where

$$m_1 = 1, \quad m_2 = 2, \quad m_3 = 5, \quad m_4 = 14.$$

Find the best possible upper bound for

$$\sum_{p \leqslant x} \frac{|\lambda_\tau(p)|}{p}$$

A optimilisation problem, 2

Answer (Tenenbaum, 2007) :

$$\sum_{p\leqslant x}\frac{|\lambda_\tau(p)|}{p}\leqslant (a\alpha+b\beta+2\gamma)\log_2 x+O(1),$$

where

$$a=\sqrt{\frac{6-\sqrt{21}}{5}},\qquad b=\sqrt{\frac{6+\sqrt{21}}{5}},$$

and

$$\alpha=\frac{102+7\sqrt{21}}{210},\qquad \beta=\frac{102-7\sqrt{21}}{210},\qquad \gamma=\frac{1}{35}.$$

Table of contents

Rankin's method, 1

- *Rankin's idea* (1985)

  Find "optimal" multiplicative functions $\lambda_{f,r}^{\pm}(n)$ such that

  $$\lambda_{f,r}^{\mp}(p^\nu) \leqslant |\lambda_f(p^\nu)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p^\nu) \quad (r \in \mathscr{R}^{\mp})$$

  for all p and $\nu \geqslant 1$, and their associated Dirichlet series

  $$\Lambda_{f,r}^{\pm}(s) := \sum_{n=1}^{\infty} \lambda_{f,r}^{\pm}(n) n^{-s}$$

  in the half-plane $\Re e\, s \geqslant 1$ is "controlled" by

  $$F_j(s) := \sum_{n=1}^{\infty} \lambda_f(n)^{2j} n^{-s} \quad (j = 1, \ldots, 4).$$

  Then tauberian theorems give Theorem 3.

Rankin's method, 2

- *Construction of* $\lambda_{f,r}^{\pm}(n)$

$$\lambda_{f,r}^{\mp}(p^\nu) := \begin{cases} \displaystyle\sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^{\mp} \lambda_f(p)^{2j} & \text{if } \ \nu = 1 \text{ and } r > 0 \\ 0 & \text{if } \ \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\mp} \\ |\lambda_f(p^\nu)|^{2r} & \text{if } \ \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\pm} \end{cases}$$

where $(a_1^{\mp}, \dots, a_4^{\mp}) \in \mathbb{R}^4$ will be choosen optimally,

$$a_0^{-} := 0 \quad \text{and} \quad a_0^{+} := 1 - a_1^{+} - a_2^{+} - a_3^{+} - a_4^{+}.$$

Rankin's method, 3

- *Dirichlet series $F_j(s)$*

  Gelbart & Jacquet (1978) : $L(s, \mathrm{sym}^2 f)$ and $L(s, \mathrm{sym}^4 f)$ are invertible for $\Re e\, s \geqslant 1$ (holomorphic for $\Re e\, s \geqslant 1$ and nonzero for $\Re e\, s = 1$).

  Kim & Shahidi (2002) : $L(s, \mathrm{sym}^6 f)$ and $L(s, \mathrm{sym}^8 f)$ are invertible for $\Re e\, s \geqslant 1$.

Lemma 1 :

For $j = 1, 2, 3, 4$ and $\Re e\, s > 1$, we have

$$F_j(s) = \zeta(s)^{m_j} G_j(s),$$

where $m_1 = 1$, $m_2 = 2$, $m_3 = 5$, $m_4 = 14$ and the $G_j(s)$ are invertible for $\Re e\, s \geqslant 1$.

Rankin's method, 4

- *Dirichlet series associated to* $\lambda_{f,r}^{\pm}(n)$

Lemma 2 :

For $r > 0$ and $\Re e\, s > 1$, we have

$$\Lambda_{f,r}^{\pm}(s) := \sum_{n=1}^{\infty} \frac{\lambda_{f,r}^{\pm}(n)}{n^s} = \zeta(s)^{\rho_r^{\pm}+1} H_{f,r}^{\pm}(s),$$

where

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4 \cdot 2a_2^{\pm} + 2^2 \cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1$$

and $H_{f,r}^{\pm}(s)$ is invertible for $\Re e\, s \geqslant 1$.

Rankin's method, 5

- *How to choose* $(a_1^{\mp}, \dots, a_4^{\mp}) \in \mathbb{R}^4$ *optimally ?*

Rankin's method, 5

- *How to choose* $(a_1^{\mp}, \dots, a_4^{\mp}) \in \mathbb{R}^4$ *optimally ?*
  - Firstly we must have

$$\lambda_{f,r}^{\mp}(p) \leqslant |\lambda_f(p)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p) \quad (r \in \mathscr{R}^{\mp})$$

for all p.

Rankin's method, 5

- *How to choose* $(a_1^{\mp}, \dots, a_4^{\mp}) \in \mathbb{R}^4$ *optimally ?*
  - Firstly we must have

$$\lambda_{f,r}^{\mp}(p) \leqslant |\lambda_f(p)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p) \quad (r \in \mathscr{R}^{\mp})$$

for all p.

  - Secondly the exponent

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4 \cdot 2a_2^{\pm} + 2^2 \cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1$$

must be maximal/minimal according to $r \in \mathscr{R}^{\mp}$.

Rankin's method, 5

- *How to choose* $(a_1^{\mp}, \dots, a_4^{\mp}) \in \mathbb{R}^4$ *optimally ?*
  - Firstly we must have

$$\lambda_{f,r}^{\mp}(p) \leqslant |\lambda_f(p)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p) \quad (r \in \mathscr{R}^{\mp})$$

for all p.

  - Secondly the exponent

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4 \cdot 2a_2^{\pm} + 2^2 \cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1$$

must be maximal/minimal according to $r \in \mathscr{R}^{\mp}$.

- *How to realize ?*

Theorical analyse + formal calculation via Maple.

Rankin's method, 6

- *Theorical analyse*

Rankin's method, 6

- *Theorical analyse*

Set $t := (|\lambda_f(p)|/2)^2$. Then $0 \leqslant t \leqslant 1$ (Deligne's inequality). Our problem can be reformulated as follows : Find polynomial

$$g_r^{\mp}(t) := \sum_{0\leqslant j\leqslant 4} a_j^{\mp} t^j$$

such that

$$g_r^{\mp}(t) \leqslant t^r \leqslant g_r^{\pm}(t) \quad (0 \leqslant t \leqslant 1,\ r \in \mathscr{R}^{\mp})$$

and the exponent

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4 \cdot 2a_2^{\pm} + 2^2 \cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1$$

must be maximal/minimal according to $r \in \mathscr{R}^{\mp}$.

Table of contents

Erdős' $\mathscr{B}$-free numbers notion

- *$\mathscr{B}$-free numbers*

  Let $\mathscr{B} = \{1 < b_1 < b_2 < \cdots\}$ be an increasing sequence of integers such that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j). \tag{4}$$

Erdős' $\mathscr{B}$-free numbers notion

- *$\mathscr{B}$-free numbers*

  Let $\mathscr{B} = \{1 < b_1 < b_2 < \cdots\}$ be an increasing sequence of integers such that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j). \tag{4}$$

  Let $\mathscr{A} = \mathscr{A}(\mathscr{B}) := \{a_i\}_{i \geqslant 1}$ (with increasing order) be the sequence of all $\mathscr{B}$-free numbers, i.e. the integers indivisible by any element in $\mathscr{B}$.

Erdős' $\mathscr{B}$-free numbers notion

- *$\mathscr{B}$-free numbers*

  Let $\mathscr{B} = \{1 < b_1 < b_2 < \cdots\}$ be an increasing sequence of integers such that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j). \tag{4}$$

  Let $\mathscr{A} = \mathscr{A}(\mathscr{B}) := \{a_i\}_{i \geqslant 1}$ (with increasing order) be the sequence of all $\mathscr{B}$-free numbers, i.e. the integers indivisible by any element in $\mathscr{B}$. Then $\mathscr{A}$ is of positive density

$$\lim_{x \to \infty} \frac{|\mathscr{A} \cap [1, x]|}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right) > 0.$$

Construction of a special $\mathscr{B}$-free numbers

- *Serre's estimate* (1981)

  If $f \in \mathrm{H}_k^*(N)$, then for all $x \geqslant 2$ and any $\delta < \frac{1}{2}$,

$$|\{p \leqslant x \,:\, \lambda_f(p) = 0\}| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}. \tag{5}$$

Construction of a special $\mathscr{B}$-free numbers

- *Serre's estimate* (1981)

  If $f \in \mathrm{H}_k^*(N)$, then for all $x \geqslant 2$ and any $\delta < \frac{1}{2}$,

$$|\{p \leqslant x \,:\, \lambda_f(p) = 0\}| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}. \tag{5}$$

- *Kohnen, Iwaniec & Sengupta's rseult* (2007)

  There is a prime $p' \ll kN^{1/2}$ such that $p' \nmid N$ and $\lambda_f(p') < 0$.

Construction of a special $\mathscr{B}$-free numbers

- *Serre's estimate* (1981)

  If $f \in \mathrm{H}_k^*(N)$, then for all $x \geqslant 2$ and any $\delta < \frac{1}{2}$,

$$|\{p \leqslant x \,:\, \lambda_f(p) = 0\}| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}. \tag{5}$$

- *Kohnen, Iwaniec & Sengupta's rseult* (2007)

  There is a prime $p' \ll kN^{1/2}$ such that $p' \nmid N$ and $\lambda_f(p') < 0$.

- *A special sequence $\mathscr{B}_f$*

$$\begin{aligned}\mathscr{B}_f = \{p : \lambda_f(p) = 0\} \cup \{p : p \mid N\} \cup \{p'\}\\ \cup \{p^2 : p \nmid p'N \text{ and } \lambda_f(p) \neq 0\}.\end{aligned}$$

Serre's (5) implies that $\mathscr{B}_f$ satisfies (4).

End of the proof of Theorem 1

- *Special $\mathscr{B}_f$-free numbers*

  Let $\mathscr{A}_f = \mathscr{A}_f(\mathscr{B}_f)$ be the sequence of all $\mathscr{B}_f$-free numbers. Then $\mathscr{A}_f$ is of positive density and $\lambda_f(a) \neq 0$ for all $a \in \mathscr{A}_f$.

End of the proof of Theorem 1

- *Special $\mathscr{B}_f$-free numbers*

  Let $\mathscr{A}_f = \mathscr{A}_f(\mathscr{B}_f)$ be the sequence of all $\mathscr{B}_f$-free numbers. Then $\mathscr{A}_f$ is of positive density and $\lambda_f(a) \neq 0$ for all $a \in \mathscr{A}_f$.

- *End of the proof of Theorem 1*

  We partition

$$\mathscr{A}_f = \mathscr{A}_f^+ \cup \mathscr{A}_f^-,$$

where $\mathscr{A}_f^\pm := \{a \in \mathscr{A}_f : \lambda_f(a) \gtrless 0\}$. Consider

$$\mathscr{N}^\pm := \mathscr{A}_f^\pm \cup \{ap' : a \in \mathscr{A}_f^\mp\}.$$

Clearly $\lambda_f(a) \gtrless 0$ and $(a, N) = 1$ for all $a \in \mathscr{N}^\pm$ and

$$\mathscr{N}_f^\pm(x) \geqslant \left|\mathscr{N}^\pm \cap [1, x]\right| \geqslant \left|\mathscr{A}_f \cap [1, x/p']\right| \qquad (x \geqslant 1).$$