EXERCISE 4

1. Define the truncation operator on functions on $\Gamma \setminus \mathbb{H}$ by

$$\Lambda^T \varphi(z) = \varphi - \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f_T(\mathrm{Im}\gamma z)$$

where

$$f_T(y) = \begin{cases} \varphi_P(y) & y > T; \\ 0 & y \leq T. \end{cases}$$

Show that
(i)

$$\Lambda^T \varphi(z) = \begin{cases} \varphi(z) & \mathrm{Im} z < T \\ \varphi(z) - \varphi_P(y) & \mathrm{Im} z > T \end{cases}$$

for $z \in \mathcal{F}$.
(ii)

$$\left(\Lambda^T \varphi_1, \varphi_2\right)_{\Gamma \setminus \mathbb{H}} = \left(\varphi_1, \Lambda^T \varphi_2\right)_{\Gamma \setminus \mathbb{H}} = \left(\Lambda^T \varphi_1, \Lambda^T \varphi_2\right)_{\Gamma \setminus \mathbb{H}}.$$

(iii)

$$\Lambda^T E(z;s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} g_T(\mathrm{Im}\gamma z),$$

where

$$g_T(y) = \begin{cases} y^s & y \leq T \\ -\phi(s) y^{1-s} & y > T \end{cases}$$

and $E(z;s)$ is Eisenstein series.

From these facts, deduce the Maass-Selberg relations.

2. Show that the Maass-Selberg relations imply

$$\left\|\Lambda^T E(z;s)\right\|^2 = \begin{cases} \frac{\mathrm{Im}\phi\left(\frac{1}{2} - \mathrm{i}t\right) T^{2\mathrm{i}t}}{t} + 2\mathrm{log}T - \frac{\phi'}{\phi}\left(\frac{1}{2} + \mathrm{i}t\right) & t \neq 0 \\ \left(2\mathrm{logT} - \phi'\left(\frac{1}{2}\right)\right)\left(1 + \phi\left(\frac{1}{2}\right)\right) & t = 0 \end{cases}$$

for $s = \frac{1}{2} + \mathrm{i}t$, $t \in \mathbb{R}$.

3. (i) Let $f \in C_c^\infty(\mathbb{R}_{>0})$. Use the approximation

$$E(z;s) - (y^s + \phi(s) y^{1-s}) \ll y^{-N} \quad y \longrightarrow \infty,$$

and integration by parts to show that the Eisenstein transform

$$E_f(z) = \frac{1}{4\pi} \int_0^\infty f(t) E(z; \frac{1}{2} + \mathrm{i}t) \mathrm{d}t$$

satisfies

$$|E_f(z)| \ll \frac{\sqrt{y}}{\mathrm{log}y} \text{ as } y \longrightarrow \infty.$$

Deduce that

$$E_f \in L^2 \left(\Gamma \setminus \mathbb{H} \right).$$

(ii) Use the Maass-Selberg formula to show that

$$\big((E_f)^T, (E_g)^T\big)_{\Gamma\setminus\mathbb{H}} = \frac{1}{(4\pi)^2}\int_0^\infty \int_0^\infty f(n)\overline{g(n')}\frac{T^{\mathrm{i}(n-n')} - T^{\mathrm{i}(n'-n)}}{\mathrm{i}(n-n')}\mathrm{d}n\mathrm{d}n' + O(\frac{1}{\log T}).$$

Deduce that

$$f \longmapsto E_f$$

extends to an isometry of $L^2(\mathbb{R}_{\geq 0}, \mathrm{d}x)$ with a subspace of $L^2\left(\Gamma \setminus \mathbb{H}\right)$.