# Langlands picture of automorphic forms and $L$-functions

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## §6 Irreducible unitary representations of $SL(2,\mathbb{R})$ (Mar. 19)

Define $W_s(\Gamma)$ as the space of all smooth functions $f$ on $\mathbb{H}$ satisfying
(1) $f$ is automorphic, i.e. $f(\gamma z) = f(z)$, $\forall \gamma \in \Gamma$;
(2) $f$ is bounded;
(3) $f$ is cuspidal;
(4) $\Delta^* f = \frac{1-s^2}{4} f$, where $\Delta^* = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

The map $f \mapsto \varphi_f(g) = f(gi)$ gives an isomorphism from $W_s(\Gamma)$ to the space of smooth functions $\varphi$ on $G = SL(2,\mathbb{R})$ satisfying
(1) $\varphi(\gamma g) = \varphi(g)$, $\forall \gamma \in \Gamma$;
(2) $\varphi$ is right $K$-invariant, i.e. $\varphi(g\kappa_\theta) = \varphi(g)$, $\forall \kappa_\theta \in K = SO(2,\mathbb{R})$;
(3) $\varphi$ is bounded;
(4) $\varphi$ is cuspidal, i.e.

$$\int_0^1 \varphi\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) g\right) \mathrm{d}x = 0$$

for almost all $g \in G$;
(5) $\Delta\varphi = \frac{1-s^2}{4}\varphi$, where

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta}.$$

**Remark 1.** In fact, since $\varphi$ is right $K$-invariant, we have $\Delta\varphi = \Delta^*\varphi$. $\Delta^*$ is self-adjoint with respect to the Petersson inner product, its eigenvalues are non-negative.

Now we know that both $S_k(\Gamma)$ and $W_s(\Gamma)$ are embedded into $\mathcal{L}_0^2(\Gamma\backslash G)$, where

$$\mathcal{L}_0^2(\Gamma\backslash G) = \{\varphi \in \mathcal{L}^2(\Gamma\backslash G), \varphi \text{ is cuspidal}\}$$

is a $G$-invariant subspace. We have the following theorem.

**Theorem 6.1.** *$\mathcal{L}_0^2(\Gamma\backslash G)$ can be decomposed as the direct sum of $G$-invariant subspaces, i.e.*

$$\mathcal{L}_0^2(\Gamma\backslash G) = \bigoplus_i H^i.$$

*Each $H^i$ is an infinite dimensional irreducible unitary representation of $G$.*

In order to prove Theorem 6.1, we need to answer the following questions.
(1) What are the irreducible unitary representations of $G = SL(2,\mathbb{R})$?
(2) Which of them occur in $\mathcal{L}^2(\Gamma\backslash G)$?

**Theorem 6.2.** *The irreducible unitary representations of $G$ are subsets of the induced representations $(\pi, H)$, $\pi = ind_B^G\chi$, where $\chi : B \to \mathbb{C}^\times$ is an irreducible one-dimensional representation of $B$ (quasi-character) given by*

$$\chi\left(\left(\begin{array}{cc} a & * \\ 0 & a^{-1} \end{array}\right)\right) = sgn(a)^\epsilon |a|^s, \qquad \epsilon = 0 \text{ or } 1.$$

*The space $H$ consists of functions $\varphi$ on $G$ satisfying*

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} g\right) = sgn(a)^{\epsilon}|a|^{s+1}\varphi(g)$$

*and*

$$\int_K |\varphi(\kappa)|^2 \, d\kappa < \infty.$$

*$\pi$ acts on $H$ as right translation, i.e.*

$$\pi(g)\varphi(h) = \varphi(hg).$$

Now we recall some facts from Lie theory.

$G = SL(2,\mathbb{R})$ is a Lie group generated by $\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ and $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Denote by $\mathfrak{g}$ the Lie algebra of $G$, which is the 3-dimensional trace zero subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Then $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}$ is generated by $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $E_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $E_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$. Let $U(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra. The Casimir element

$$\triangle = \frac{1}{4}\left(W^2 - \frac{E_+E_-}{2} - \frac{E_-E_+}{2}\right)$$

is in the center of $U(\mathfrak{g}_{\mathbb{C}})$. It agrees with the Laplacian operator when acts on smooth functions.

Suppose that $(\pi, V)$ is an irreducible unitary representation of $G$, where $V$ is complex Hilbert space. For $X \in \mathfrak{g}$, $\varphi \in V$, the endomorphism

$$(\mathrm{d}\pi(X))(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\pi(e^{tX})\varphi\right)\bigg|_{t=0} = \lim_{t\to 0}\frac{1}{t}\left(\pi(e^{tX})\varphi - \varphi\right)$$

gives the Lie algebra representation, called the *differential* of the Lie group representation $\pi$. We call $\varphi \in V$ is $\mathcal{C}^1$, if $\mathrm{d}\pi(X)$ is defined, and

$$g \mapsto \pi(g)\left(\mathrm{d}\pi(X)f\right)$$

is a continuous function of $G$. We call $\varphi$ is $\mathcal{C}^k$, if $\varphi$ is $\mathcal{C}^1$ and $\mathrm{d}\pi(X)\varphi$ is $\mathcal{C}^{k-1}$ for all $X \in \mathfrak{g}$. We call $\varphi$ is $\mathcal{C}^\infty$, if $f$ is $\mathcal{C}^k$ for all $k \geq 1$. Since $\triangle$ is in the center of $U(\mathfrak{g}_{\mathbb{C}})$, $\triangle$ acts on $V$ as scalar, i.e.

$$\triangle\varphi = \lambda\varphi, \quad \text{for } \varphi \in H.$$

Now we study the induced representation $(\pi, H)$ in theorem 6.2. It is easy to check that $\Delta$ acts on smooth function in $H$ as scalar $\lambda$. The basic idea is to restrict the representation to a maximal compact subgroup of $G$ and consider the action of $\mathfrak{g}_{\mathbb{C}}$. Since $K = SO(2,\mathbb{R})$ is a maximal compact subgroup of $G$, and $K$ is abelian, all the irreducible unitary representations of $K$ are one-dimensional, given by

$$\kappa_\theta \longmapsto e^{in\theta}, \qquad \text{for some } n \in \mathbb{Z}.$$

Therefore, one has

$$H = \bigoplus_{n\in\mathbb{Z}} H_n,$$

where $H_n = \{\varphi \in H : \pi(\kappa_\theta)\varphi = e^{in\theta}\varphi\}$.

**Remark 2.** Since $\pi\left((-I)^2\right) = \pi(I) = 1$, we have $\pi(-I) = (-1)^\epsilon$, $\epsilon = 0$ or 1. For $\varphi \in H_n$,

$$\pi(-I)\varphi = \pi\left(\begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix}\right)\varphi = e^{in\pi}\varphi = (-1)^\epsilon\varphi.$$

Thus $H_n$ is a zero space if $n \not\equiv \epsilon(\mathrm{mod} 2)$. In fact, one has the following proposition.

**Proposition 6.3.** *Assume that $H_n \neq 0$. We have*
*1) $dim H_n = 1$, and $H_n$ is a $\mathcal{C}^\infty$-vector space.*
*2) $H_n$ is the eigenspace of $d\pi(W)$ with eigenvalue in.*
*3) $d\pi(E_\pm) : H_n \to H_{n\pm 2}$, i.e., for $0 \neq \varphi \in H_n$,*

$$d\pi(E_\pm)\varphi_n = (s + 1 \pm n)\varphi_{n\pm 2}.$$

Since $\triangle$ acts on $H$ as multiplication, i.e. $\triangle\varphi = \lambda\varphi$ with $\lambda = \frac{1-s^2}{4}$ and $\varphi \in H$. By Proposition 6.3, if $s$ satisfies

$$s + 1 \equiv \epsilon \mod 2, \qquad s + 1 = m \text{ for some } 0 < m \in \mathbb{Z}, \tag{6.1}$$

then $H_m$ is annihilated by $d\pi(E_-)$, and $H_{-m}$ is annihilated by $d\pi(E_+)$. Thus we have the following two $\mathfrak{g}_\mathbb{C}$-invariant subspaces

$$\bigoplus_{\substack{n\equiv\epsilon(\bmod 2)\\ n\geq m}} H_n \quad \text{and} \quad \bigoplus_{\substack{n\equiv\epsilon(\bmod 2)\\ n\leq -m}} H_n.$$

If $s$ does not satisfy (6.1), we have

$$\bigoplus_{n\equiv\epsilon(\bmod 2)} H_n$$

is $\mathfrak{g}_\mathbb{C}$ invariant and has no non-trivial $\mathfrak{g}_\mathbb{C}$-invariant subspaces.

Based on the facts above, we can give a classification of the infinitesimal equivalent classes of irreducible representations of $G$. And then we consider the relation between infinitesimal equivalent classes and the equivalent classes of the irreducible unitary representations of $G$. We omit the details here, but only give the classification of the irreducible unitary representations.

According theorem 6.2, the irreducible unitary representations of $G$ are given as following.

**Case 1.** $s = \mathrm{i}t \in \mathrm{i}\mathbb{R}$
In this case $\chi$ is unitary. Via knowledge in representation theory, a representation induced from a unitary representation is unitary. Therefore $\pi$ is unitary.

If $s \neq 0$, $s$ does not satisfy (6.1). Thus $\pi$ is an irreducible unitary representation. We denote it by $\pi^{\mathrm{i}t,\epsilon}$ and call it *principal series*. The representation space has decomposition

$$\bigoplus_{n\equiv\epsilon(\bmod 2)} H_n.$$

If $s = 0$ and $\epsilon = 0$, then $\chi$ is the trivial character.

If $s = 0$ and $\epsilon = 1$, then we know $m = 1$ is a solution of (6.1). $\pi^{0,1}$ decomposes as sum of two irreducible unitary representation of $G$, i.e.

$$\pi^{0,1} = D_1^+ \bigoplus D_1^-,$$

where $D_1^+$ and $D_1^-$ have representation space

$$\bigoplus_{\substack{n\equiv\epsilon \\ n\geq 1(\bmod 2)}} H_n \qquad \text{and} \qquad \bigoplus_{\substack{n\equiv\epsilon \\ n\leq -1(\bmod 2)}} H_n,$$

respectively. $D_1^+$ and $D_1^-$ are called *the limit of the discrete series.*

**Case 2.** $0 \neq s \in \mathbb{Z}$
If $0 \neq s \in \mathbb{Z}$ and $s+1 \equiv \epsilon \bmod 2$, there exists $m = s+1$ satisfying (6.1) and $\pi^{s,\epsilon}$ is not unitary. However, it contains two sub(or quotient) irreducible unitary representations $D_{s+1}^+$ and $D_{s+1}^-$ of which representation spaces are

$$\bigoplus_{\substack{n\equiv\epsilon \\ n\geq s+1(\bmod 2)}} H_n \qquad \text{and} \qquad \bigoplus_{\substack{n\equiv\epsilon \\ n\leq -(s+1)(\bmod 2)}} H_n,$$

respectively. Such irreducible unitary representation are called *Discrete series.* For example,

$$\pi^{1,0} \supseteq D_2^+ \bigoplus D_2^- \quad \text{and} \quad \pi^{1,0}/\left(D_2^+ \bigoplus D_2^-\right) = H_0,$$
$$\pi^{-1,0}/H_0 = D_2^+ \bigoplus D_2^-.$$

and

$$\pi^{2,1} \supseteq D_3^+ \bigoplus D_3^- \quad \text{and} \quad \pi^{2,1}/\left(D_3^+ \bigoplus D_3^-\right) = H_1 \bigoplus H_{-1},$$
$$\pi^{-2,1}/\left(D_3^+ \bigoplus D_3^-\right) = H_1 \bigoplus H_{-1}.$$

**Case 3.** $-1 < s < 1$ and $\epsilon = 0$
In this case, $\chi$ is not unitary. However, $\pi^{s,0}$ is irreducible unitary representations. These representations are called *complement series.*

**Proposition 6.4.** *The above argument gives the irreducible unitary representations as subset of* $\pi = \mathrm{ind}_B^G \chi$. *They are*

1. *the principle series* $\pi^{it,\epsilon}$ *with* $t \neq 0$*;*
2. *the complement series* $\pi^{s,0}$ *with* $\epsilon = 0$ *and* $-1 < s < 1$, $s \neq 0$*;*
3. *the discrete series* $D_{s+1}^+$ *and* $D_{s+1}^-$, *where* $0 < s \in \mathbb{Z}$ *and* $s+1 \equiv \epsilon \bmod 2$*;*
4. *the limit of discrete series* $D_1^+$ *and* $D_1^-$, *with* $\epsilon = 1$ *and* $s = 0$*;*
5. *the one-dimensional trivial representation* $1$ *with* $\epsilon = 0$ *and* $s = 0$.

The irreducible unitary representations have a deep relation with the Casimir element(Laplace operator) $\Delta$. One can use the eigenvalue of $\Delta$ to classify which irreducible representation occurs in $L^2(\Gamma\backslash G)$.

$W_s(\Gamma)$ and $S_k(\Gamma)$ can be embedded in $L_0^2(\Gamma\backslash G) = \bigoplus_i H^i$. This implies the discrete series $D_k^+$ and $D_k^-$ with $\epsilon = 0$ and the principle or complement series $\pi^{s,0}$ where $s$ satisfies

$$\lambda = \frac{1-s^2}{4}, \qquad \lambda \text{ is the eignvalue of } \Delta^* \text{ on } W_s(\Gamma)$$

occur in $L_0^2(\Gamma\backslash G)$. Selberg conjectured that the complement series $\pi^{s,0}$ (i.e. $-1 < s < 1$) never occurs for congruence group $\Gamma_0(N)$, i.e. $\lambda > \frac{1}{4}$. In fact, he proved the following theorem.

**Theorem 6.5.** *For $\Gamma = SL_2(\mathbb{Z})$, we have $\lambda = \frac{1-s^2}{4} \geq \frac{3\pi}{2}$; for $\Gamma$ a general congruence subgroup, we have $\lambda \geq \frac{3}{16}$.*

**Remark 3.** The above result was improved to $\lambda > \frac{3}{16}$ by Jacquet - Gelbart in 1976 by method of the Gelbart-Jacquet lift up to $GL_3$. Shahidi-Kim-Sanark have a better result by lifting up to $GL_4$ and $GL_5$.