Langlands picture of automorphic forms and L-functions

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§1 Automorphic forms on $GL(1)$ (Mar. 3)

First, we list some references for this lecture as follows, ordered by published time: Tate (1950) [Ta], Goldstein (1971) [Go], Gelbart (1975) [Ge], Bump (1997) [Bu], Ramakrishnan and Valenza (1999) [RV], Kulda (2003) [Ku].

Theorem 1.1 (**Riemann, 1859**). *The meromorphic function* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ *defined for* $\mathrm{Re}(s) > 1$ *extends analytically to all of* $\mathbb{C}$ *and satisfies*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

This theorem appears in Riemann's nine page paper in 1859, which is one in Number theory. In fact, it shows that a large part of L-functions are just Mellin transform of theta functions. In 1910's Hecke gave his remarkable work in Number theory, and in 1950, Tate [Ta] gave another method, i.e. harmonic analysis to get the functional equations of L-functions.

Lemma 1.2. *For* $\mathrm{Re}(s) > 1$, $\zeta(s)$ *is an analytic function.*

Proof. For $s = \sigma + it$ and $\sigma \geqslant \sigma_0 > 1$, we have

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{\sigma}} \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}} < 1 + \int_1^{\infty}\frac{\mathrm{d}u}{u^{\sigma_0}} = 1 + \frac{1}{\sigma_0 - 1}.$$

Then by Weierstrass this shows $\zeta(s)$ is holomorphic. □

Theorem 1.3 (**Euler**). *For* $\mathrm{Re}(s) > 1$, *we have*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}. \tag{1.1}$$

Proof. Let $\zeta_P(s) = \prod_{p \leqslant P}(1 - p^{-s})^{-1}$. We need to show $\lim_{P\to\infty}\zeta_P(s) = \zeta(s)$. Since for $\mathrm{Re}(s) > 1$,

$$(1 - p^{-s})^{-1} = 1 + p^{-s} + p^{-2s} + \cdots,$$

we have

$$\prod_{p\leqslant P}(1 - p^{-s})^{-1} = \prod_{p\leqslant P}\sum_{m=0}^{\infty} p^{-ms} = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \cdots$$

where $n_1, n_2, \cdots$ are those integers none of whose prime factors exceed P. The fundamental theorem of arithmetic (FTA) says that all the integers up to P are of this form. Thus

$$\begin{aligned}\left|\zeta(s) - \prod_{p\leqslant P}(1 - p^{-s})^{-1}\right| &= \left|\zeta(s) - \left(1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \cdots\right)\right| \\ &\leqslant \frac{1}{(P+1)^{\sigma}} + \frac{1}{(P+2)^{\sigma}} + \cdots.\end{aligned}$$

This tends to 0 as $P \to \infty$, if $\mathrm{Re}(s) > 1$; thus (1.1) follows. $\square$

Corollary 1 (Euler-Euclid). *There exist infinitely many primes.*

Proof. Let $s = 1$ in (1.1). It immediately follows from the divergence of the left hand side that there exists infinitely many primes on the right hand side. $\square$

Corollary 2. *For* $\mathrm{Re}(s) > 1$, $\zeta(s) \neq 0$.

Proof. We have for $\sigma = \mathrm{Re}(s) > 1$

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\cdots\left(1 - \frac{1}{P^s}\right)\zeta(s) = 1 + \frac{1}{m_1^s} + \frac{1}{m_2^s} + \cdots,$$

where $m_1, m_2, \cdots$ are the integers all of whose prime factors exceed P. Hence

$$\left|\left(1 - \frac{1}{2^s}\right)\cdots\left(1 - \frac{1}{P^s}\right)\zeta(s)\right| \geqslant 1 - \frac{1}{(P+1)^\sigma} - \frac{1}{(P+2)^\sigma} - \cdots > 0$$

if P is large enough. Hence $|\zeta(s)| > 0$. $\square$

Proof of Theorem 1.1. Recall

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-t} t^s \frac{\mathrm{d}t}{t}$$

for $\mathrm{Re}(s) > 1$. Then we have

$$\begin{aligned}
\pi^{-s}\Gamma(s)\zeta(2s) &= \sum_{n=1}^\infty \int_0^\infty (\pi n^2)^{-s} t^s \mathrm{e}^{-t}\frac{\mathrm{d}t}{t} \\
&\overset{t\to\pi n^2 t}{=} \sum_{n=1}^\infty \int_0^\infty t^{s-1}\mathrm{e}^{-\pi n^2 t}\mathrm{d}t \\
&= \int_0^\infty t^{s-1}\left(\theta(\mathrm{i}t) - \frac{1}{2}\right)\mathrm{d}t \\
&= \int_1^\infty t^{s-1}\left(\theta(\mathrm{i}t) - \frac{1}{2}\right)\mathrm{d}t - \frac{1}{2}\left(\frac{t^s}{s}\right)\Big|_0^1 + \int_0^1 t^s\theta(\mathrm{i}t)\frac{\mathrm{d}t}{t} \\
&= \int_1^\infty t^{s-1}\left(\theta(\mathrm{i}t) - \frac{1}{2}\right)\mathrm{d}t - \frac{1}{2s} + \int_1^\infty t^{-s-1}\theta\left(\frac{\mathrm{i}}{t}\right)\mathrm{d}t \qquad (1.2)
\end{aligned}$$

where $\theta(\mathrm{i}t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t}$. Here $\theta(\mathrm{i}t)$ is the special case of $\theta(\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\pi \mathrm{i} n^2\tau}$ for $\mathrm{Re}(\tau) = 0$. We assert

$$\theta\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{\mathrm{i}}\right)^{\frac{1}{2}}\theta(\tau).$$

But we'll only use and prove later the following special case which is still called the automorphy of θ-function:

$$\theta\left(\frac{\mathrm{i}}{t}\right) = t^{\frac{1}{2}}\theta(\mathrm{i}t). \qquad (1.3)$$

Substituting (1.3) to (1.2), we obtain

$$\pi^{-s}\Gamma(s)\zeta(2s) = \int_1^\infty (t^{s-1} + t^{-s-1/2})\left(\theta(\mathrm{i}t) - \frac{1}{2}\right)\mathrm{d}t - \frac{1}{2s} - \frac{1}{1-2s}.$$

Taking $s \to s/2$, we get

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^{\infty}(t^{s/2-1}+t^{-s/2-1/2})\left(\theta(\mathrm{i}t)-\frac{1}{2}\right)\mathrm{d}t - \frac{1}{s(s-1)}. \tag{1.4}$$

The assertion of the theorem follows immediately from the above equation.

Note that

$$\theta(\mathrm{i}t)-\frac{1}{2} = \sum_{n=1}^{\infty}\mathrm{e}^{-\pi n^2 t} < \sum_{n=1}^{\infty}\mathrm{e}^{-\pi n t} = \frac{\mathrm{e}^{-\pi t}}{1-\mathrm{e}^{-\pi t}} = O(\mathrm{e}^{-\pi t}),$$

we can obtain that the integral in (1.4) converges and thus $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is analytic on $\mathbb{C}$ except $s = 0, 1$. This says $\zeta(s)$ is non-zero for $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s) < 0$ and $s \neq -2m, m \in \mathbb{Z}$.

Now all that is left to be done is to establish (1.3). In view of this, we'll use the following Poisson summation formula (PSF), for "nice" f,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty}\mathrm{e}^{2\pi \mathrm{i}kx}\int_{-\infty}^{\infty} f(x_1)\mathrm{e}^{-2\pi\mathrm{i}kx_1}\mathrm{d}x_1.$$

Take

$$f(x) = \mathrm{e}^{-tx^2} \quad (t > 0).$$

We compute

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{e}^{-tx^2+2xy}\mathrm{d}x &\overset{x\to\frac{1}{\sqrt{t}}x}{=} \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2+2xy/\sqrt{t}}\mathrm{d}x \\ &= \frac{\mathrm{e}^{y^2/t}}{\sqrt{t}}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-y/\sqrt{t})^2}\mathrm{d}x \\ &= \sqrt{\frac{\pi}{t}}\,\mathrm{e}^{y^2/t}.\end{aligned}$$

Taking $y \to \pi \mathrm{i}y$, we get

$$\int_{-\infty}^{\infty}\mathrm{e}^{-tx^2+2\pi\mathrm{i}xy}\mathrm{d}x = \sqrt{\frac{\pi}{t}}\,\mathrm{e}^{-\pi^2y^2/t}.$$

Then by PSF, we have

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-t(x+n)^2} = \sqrt{\frac{\pi}{t}}\sum_{k=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}kx-\pi^2k^2/t}.$$

Taking $x = 0$, we obtain

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-tn^2} = \sqrt{\frac{\pi}{t}}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\pi^2k^2/t},$$

and then $t \to \pi t$,

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\pi tn^2} = \sqrt{\frac{1}{t}}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\pi k^2/t}$$

which establish (1.3). □

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