Manifolds and Lie Groups
Final Exam, Spring 2016

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Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two hours to complete this test. Good luck!*

—— 14:00–16:00, July 1, 2016.

1. Prove that if $(U, x_1, \cdots, x_d)$ is a coordinate system on a manifold M, then $[\partial/\partial x_i, \partial/\partial x_j] = 0$ on U.

2. Prove that any smooth vector field on a compact manifold is complete.

3. Let $\omega_1, \cdots, \omega_k$ be 1-forms on a smooth n-dimensional manifold M. Show that $\{\omega_i\}_{i=1}^k$ are linearly independent if and only if $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$.

4. For any $X, Y \in \mathfrak{gl}(n, \mathbb{C})$, show that, for t near 0

(1), $\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$;

(2),

$$\lim_{n\to\infty}\left(\exp\frac{X}{n}\exp\frac{Y}{n}\right)^n = \exp(X+Y).$$

5. Let G and H be Lie groups, and $\phi : G \to H$ a Lie group homomorphism. Prove:

(1) The map $d\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, i.e. for any left invariant vector fields $X, Y \in \mathfrak{g}, d\phi([X,Y]) = [d\phi(X), d\phi(Y)]$.

(2) exp is natural, i.e. $\phi \circ \exp = \exp \circ d\phi$.

6. Suppose G is a connected Lie group, with Lie algebra $\mathfrak{g}$. Define their centers by $Z(G) = \{g \in G \,|\, gh = hg, \ \forall h \in G\}$ and $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X,Y] = 0, \ \forall Y \in \mathfrak{g}\}$, respectively. Show

(1) $Z(G)$ is a Lie subgroup of G.

(2) $Z(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{g}$.

(3) The Lie algebra of $Z(G)$ equals $Z(\mathfrak{g})$.