# Differential Geometry, Fall 2014
## School of Mathematics, Shandong University

注意事项:
A, 卷面分5分,试题总分95分.其中卷面整洁,书写规范(5分);
卷面较整洁,书写较规范(3分);书写潦草,乱涂乱画(0分).
B, 可能用到的公式:

1, $\Gamma_{ij}^k = \frac{1}{2}\sum g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right)$.

2, $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right), \quad (a>b).$

14:00–16:30, Jan. 20, 2015

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***1.*(15 points) (1), Find the curvature and torsion of** $\alpha(t) = (\cos t, \sin t, 3t)$.

**(2), Suppose** $\gamma$ **is an arc length parametrized curve with the property that**

$$|\gamma(s)| \leq |\gamma(s_0)| = R$$

**for all** $s$ **sufficiently close to** $s_0$**. Prove that the curvature** $k(s_0) \geq \frac{1}{R}$.

***2.*(10 points) Suppose** $\mathtt{x}$ **is a coordinate patch such that** $g_{11} = 1$ **and** $g_{12} = 0$**. Prove that the** $u^1$**-curve are geodesic.**

***3.*(20 points) Let** $X_N$ **be the tangential component of the normal vector** $N$ **of a unit speed curve** $\gamma$ **on a surface** $M$**. Let** $\mathtt{n}$ **be the unit normal vector to a coordinate patch in** $M$

**(1), Prove that** $X_N = N - \langle N, \mathtt{n}\rangle \mathtt{n}$ **and** $X_N$ **is a vector field along** $\gamma$**.**

**(2), Prove that the following are equivalent:**

**(i),** $X_N = 0$**; (ii),** $\gamma$ **is a geodesic; (iii),** $X_N$ **is parallel along** $\gamma$**.**

***4.*(20 points) (1), State the local Gauss-Bonnet formula.**

**(2), Let** $\mathtt{x}(u,v) = (\cos u\cos v, \cos u\sin v, \sin u)$ **be the unit sphere. Let** $\mathcal{R}$ **be the region bounded by the meridians** $v = 0, \frac{\pi}{2}$ **and the circles of latitude** $u = 0, \frac{\pi}{4}$**. Checking the local Guass-Bonnet formula for the region** $\mathcal{R}$**.**

***5.*(30 points) Consider the Torus** $T$ **parametrized by** $\mathtt{x} : [0, 2\pi]^2 \to \mathbb{R}^3$ **with**

$$\mathtt{x}(u,v) = ((a+\cos u)\cos v, (a+\cos u)\sin v, \sin u)\,,\ a > 1.$$

**(1), Compute the first and second fundamental forms.**

**(2), Compute the Gaussian curvature** $K$ **and the mean curvature** $H$**.**

**(3), Find the elliptic, hyperbolic and parabolic points.**

**(4), Checking the global Gauss-Bonnet formula for the torus $T$:**

$$\iint_T K dA = 2\pi\chi(T).$$

**(5), Show that the Willmore inequality:**

$$\iint_T H^2 dA \geq 2\pi^2.$$

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