Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§9 Application of the Selberg Trace Formula (March 9, 2009)

Last time, we showed Weyl law :

$$\sharp\{t_j \le T\} - \int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2}+ir)dr \sim \frac{\text{area}(\Gamma\backslash\text{H})}{4\pi}T^2,$$

where

$$\int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2}+ir)dr$$

equals to the number of poles of ϕ with $|\text{Im}| \le T$. If Γ is a congruence subgroup, then ϕ is a product of $\text{L}(s,\chi)^{\pm 1}$, where χ is Dirichlet character, and then

$$\int_{-T}^{T} \frac{\phi'}{\phi}(\frac{1}{2}+ir)dr \ll T\log T,$$

and therefore

$$\sharp\{t_j \le T\} \sim \frac{\text{area}(\Gamma\backslash\text{H})}{4\pi}T^2. \tag{0.1}$$

For other Γ we do not expect (0.1), because the number of poles of ϕ with $|\text{Im}| \le T$ is large than δT^2, for some $\delta > 0$ and large T (This is supposed to happen for "generic Γ", but there is no unconditional proof). Then RH is not true for ϕ, i.e. $\text{Re}(\eta)$ will concentrate near $\frac{1}{2}$.

Recall, for $\Gamma = \text{SL}_2(z)$, we have

$$\phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)}.$$

And RH $\Rightarrow$ poles of $\phi(s)$ are on $\text{Re}(s) = \frac{1}{4}$ except $s = 1$.

For generic Γ, $\text{Re}(\eta)$ concentrate near $\frac{1}{2}$, and

$$\sum_{\eta_n \text{ pole of } \phi} \frac{\frac{1}{2} - \text{Re}(\eta_n)}{|\eta_n|^2} < \infty.$$

If $\eta_n \ll \sqrt{n}$, then $\frac{1}{2} - \text{Re}\eta_n$ is not bounded away from 0.

The Weyl law is a consequence of the Selberg trace formula where the test function is localized near 0.

Today, we will estimate the number of primitive hyperbolic classes, weighted by their length.

Selberg Trace Formula :

$$\sum_{j=0}^{\infty} h(t_j) - \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)\mathrm{d}r$$
$$= \frac{\text{area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h(r)r\tanh\pi r\mathrm{d}r + 2\sum_{P}\sum_{l=1}^{\infty}\frac{g(l\log p)}{p^{\frac{l}{2}}-p^{-\frac{l}{2}}}\log p$$
$$+ \text{ other explict forms.}$$

We would like (following Selberg) a "Prime Number Theorem for primitive hyperbolic classes(= closed geodesics). We have showed that $\log p =$ length of closed geodesic. Take

$$g(x) = \frac{e^{-\frac{x}{2}} + e^{\frac{x}{2}}}{2}q(x),$$

where q is smooth, even and supported in $|x| \leq \log(X+Y)$, and $q \equiv 1$ for $|x| \leq \log(X)$. Graph of q is

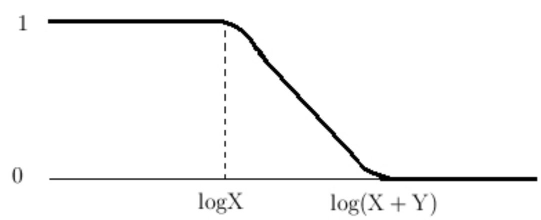

FIGURE 1

What is $h(t)$?
Firstly, on $\Re s = 1/2$, we have

$$h(t) = \int_{-\infty}^{+\infty} g(x)e^{itx}\mathrm{d}x = \frac{1}{2}\int_{0}^{\log(X+Y)} (e^{x/2}+e^{-x/2})e^{itx}q(x)\mathrm{d}x.$$

We consider the symmetric function having the form

$$\int_{0}^{+\infty} (e^{sx}+e^{(1-s)x})q(x)\mathrm{d}x,$$

since $q(x)$ is even.

Using integration by parts once, we get the trivial bound as follows:

$$\begin{aligned}\int_0^{+\infty} e^{sx} q(x)\mathrm{d}x &= \int_0^{+\infty} \frac{e^{sx}}{s} q'(x)\mathrm{d}x \\ &\ll \frac{\sqrt{X+Y}}{|s|}\int_0^{+\infty} |q'(x)|\mathrm{d}x \\ &= \frac{\sqrt{X+Y}}{|s|} \\ &\ll \frac{\sqrt{X}}{|s|}. \end{aligned} \tag{0.2}$$

If integrate three times by parts, we get the following

$$\begin{aligned}\int_0^{+\infty} \frac{e^{sx}}{s^3} q'''(x)\mathrm{d}x &\leq \frac{1}{|s|^3}\sqrt{X}\left(\log(X+Y)-\log X\right)^{-3} \\ &\times \int_0^{\log(X+Y)-\log X} \phi'''\Big(\frac{t}{\log(X+1)-\log X}\Big)\mathrm{d}t \\ &\ll \frac{1}{|s|^3}\sqrt{X}\left(\log(X+Y)-\log X\right)^{-2} \\ &\ll \frac{1}{|s|^3}\sqrt{X}T^2, \end{aligned} \tag{0.3}$$

where we set $T=\frac{X}{Y}$. By (0.2) and (0.3), we deduce the bound of $h(t)$

$$h(t) \ll \frac{\sqrt{X}}{|s|}\min\left\{1,\ \frac{T^2}{|s|^2}\right\}. \tag{0.4}$$

Now for $\frac{1}{2}<s\leq 1$, we have

$$\begin{aligned}\int_0^{+\infty} e^{sx} q(x)\mathrm{d}x &\ll \int_0^{\log X} e^{sx}\mathrm{d}x + \int_{\log X}^{\log(X+Y)} e^{sx} q(x)\mathrm{d}x \\ &= \frac{X^s}{s} + \int_{\log X}^{\log(X+Y)} e^{sx}\mathrm{d}x \\ &= \frac{X^s}{s} + O\Big(\frac{(X+Y)^s - X^s}{s}\Big) \\ &= \frac{X^s}{s} + O(Y). \end{aligned} \tag{0.5}$$

This because

$$\begin{aligned} \frac{(X+Y)^s - X^s}{s} &= \int_X^{X+Y} t^{s-1}\mathrm{d}t \\ &\ll Y(X+Y)^{s-1} \\ &\ll YX^{s-1} \ll Y. \end{aligned}$$

On the other hand, we have

$$\int_0^{+\infty} e^{(1-s)x} q(x)\mathrm{d}x \le \int_0^{\log(X+Y)} e^{\frac{1}{2}x}\mathrm{d}x \ll \sqrt{X}. \tag{0.6}$$

For the spectral side, we have

$$\sum_{\frac{1}{2}<s_j\le 1} h(t_j) + \sum_{t_j\in\mathbb{R}} h(t_j) - \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)\mathrm{d}r. \tag{0.7}$$

Using (0.4), for the second term of (0.7), we obtain

$$\begin{aligned} \sum_{t_j\in\mathbb{R}} h(t_j) &\le \sum_{t_j\le T} \frac{\sqrt{X}}{|s_j|} + \sum_{t_j>T} \frac{\sqrt{X}}{|s_j|^3} T^2 \\ &\ll \sqrt{X}\Big(\sum_{j\le T^2} \frac{1}{\sqrt{j}} + T^2 \sum_{j>T^2} \frac{1}{j^{3/2}}\Big) \\ &\ll \sqrt{X}T. \end{aligned}$$

For the continuous part, using (0.4) again, we get

$$\begin{aligned} &-\frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)\mathrm{d}r \\ \ll\ & -\int_0^T \frac{\sqrt{X}}{1+r}\frac{\phi'}{\phi}(\frac{1}{2}+ir)\mathrm{d}r - \sqrt{X}T^2\int_T^{+\infty} \frac{\frac{\phi'}{\phi}(\frac{1}{2}+ir)}{r^3}\mathrm{d}r. \end{aligned} \tag{0.8}$$

We having the following formula

$$\int_{-t}^{+t} \frac{\phi'}{\phi}(\frac{1}{2}+ir)\mathrm{d}r \ll t^2. \tag{0.9}$$

For the second term of (0.8), using partial integrating and (0.9), we obtain

$$\int_T^{+\infty} \frac{\int_0^r \frac{\phi'}{\phi}(\frac{1}{2}+it)\mathrm{d}t}{r^4}\mathrm{d}r \ll \int_T^{+\infty} \frac{\mathrm{d}r}{r^2} = \frac{1}{T},$$

while for the first term, we get

$$\sqrt{X}\int_1^T \frac{\int_1^r \frac{\phi'}{\phi}(\frac{1}{2}+it)\mathrm{d}t}{r^2}\mathrm{d}r \ll \sqrt{X}T.$$

From (0.5) and (0.6), we get

$$\sum_{\frac{1}{2}<s_j\leq 1} h(t_j) = \sum_{\frac{1}{2}<s_j\leq 1} \frac{x^{s_j}}{s_j} + O(Y+\sqrt{X}).$$

So far, for the test function $g(x) = (e^{x/2}+e^{-x/2})q(x)$, we have from the spectral side

$$= \sum_{\frac{1}{2}<s_j\leq 1} \frac{x^{s_j}}{s_j} + O(Y+\sqrt{X}T).$$

In the geometric side, so only the case $l=1$ matters. We need to compute

$$\sum_{NP=p} \tanh\big(\frac{\log p}{2}\big)\log p\ q(\log p) = \sum_p \log p q(\log p) + O(Y+\sqrt{X}T).$$

The identity contribution

$$\begin{aligned}\int_{-\infty}^{+\infty} h(t)t\tanh(\pi t)\mathrm{d}t &< \int_0^T \frac{\sqrt{X}}{t}\mathrm{d}t + T^2\int_T^{+\infty}\frac{\sqrt{X}t}{t^2}\mathrm{d}t\\ &\ll \sqrt{X}T.\end{aligned}$$

And the elliptic, parabolic cases are easy to bound within error term of $O(Y+\sqrt{X}T)$.

All in all, we have

$$\sum_P q(\log p)\log p = \sum_{\frac{1}{2}<s_j\leq 1} \frac{X^{s_j}}{s_j} + O(Y+\sqrt{X}T), \tag{0.10}$$

replace X by $X\pm Y$, and subtract, we get

$$\begin{aligned}\sum_{X<p\leq X+Y} q(\log p)\log p &\leq \text{difference LHS} = \text{difference RHS}\\ &= \sum_{\frac{1}{2}<s_j\leq 1} \frac{(X+Y)^{s_j}-X^{s_j}}{s_j} + O(Y+\sqrt{X}T).\end{aligned}$$

Going back

$$\sum_{p\leq X}\log p = \sum_{\frac{1}{2}<s_j\leq 1}\frac{X^{s_j}}{s_j} + O(Y+\sqrt{X}T)$$

The minimum of the error term $O(\sqrt{X}T=Y)$ is obtained when $Y=X^{3/4}$. We have the following:

Theorem (Selberg). *For* $X\geq 1$,

$$\sum_{p\leq X}\log p = \sum_{\frac{3}{4}<s_j\leq 1}\frac{X^{s_j}}{s_j} + O(X^{3/4}). \tag{0.11}$$

For any congruence subgroup, we know that $s \leq \frac{3}{4}$ for all $\frac{1}{2} \leq s \neq 1$. Thus

$$\sum_{p \leq X} \log p = X + O(X^{3/4}).$$

This is the prime number theory with an error term. The Selberg zeta-function satisfies an analogue of the Riemann hypothesis, at least for $\Gamma = SL_2(\mathbb{Z})$. Luo-Sarnak: Can improve (0.11) to $\mathcal{O}(X^{\frac{7}{10}+\varepsilon})$ for $\Gamma = SL_2(\mathbb{Z})$.

We have following relations: hyperbolic conjugacy classes for $\Gamma = SL_2(\mathbb{Z}) \leftrightarrow 1 \neq$ units in orders of quadratic real fields $\mathbb{Q}(\sqrt{d})$ (orders are indexed by $d \equiv 0, 1(\text{mod} 4)$, $d > 0$ and not a square. In each order, the group of units is $\cong Z$), with multiplicity h_d = class number of the order ($\sigma_c \sim bc \Leftrightarrow \sigma_c = cb$, where $c \in \mathbb{Q}(\sqrt{d})$, Norm $c = 1$) $\leftrightarrow$ indefinite binary quadratic form not split over $\mathbb{Q}$.

The result of Luo-Sarnak had a connection with indefinite binary quadratic forms, since the lengths of closed geodesics on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ are given by $\log \varepsilon_d^2$ with multiplicity h_d, where ε_d is the fundamental unit (for non primitived h_d is determined by h_d, where $d = d_0 t^2$).

As in the ordinary prime number theorem,

$$\sum_{n \leq x} \Lambda(n) \sim x \Leftrightarrow \pi(x) \sim \text{Li} x.$$

So we got

$$\sum_{\log \varepsilon_d \leq X} h_d \log \varepsilon_d = X + O(X^{\frac{7}{10}+\varepsilon}).$$

In Sarnak's thesis: last formula is equivalent to

$$\sum_{\varepsilon_d^2 \leq X} h_d \sim \text{Li} X + O(X^{\frac{7}{10}+\varepsilon}),$$

and

$$\sum_{\log \varepsilon_d \leq X} h_d \log \varepsilon_d \sim * X^{3/2}.$$

For negative discrimiant d, we have $\sum_{-d \leq X} h_d \sim * X^{3/2}$.

We do not know much about the distribution of h_d by $d \leq X$, For instance, we do not know whether $h_d = 1$ infinitely often.

Go back to Weyl Law.

$$\#\{t_j \leq T\} = \frac{\text{area}(\Gamma\backslash\text{H})}{T}^2 + cT \log T + O(T).$$

Can we improve the error term?
This is important in order to get bounds on $m(\lambda)$, where $m(\lambda)$ is the multiplicities of eigenvalues, also the dimension of eigenspaces. It is conjectured that $m(\lambda) = 1$ for $\Gamma = \text{SL}_2(z)$.

We know the following bound for $\varepsilon = 1$

$$m(\lambda) \leq \#\{t_j : |t_j - t| < \varepsilon\} \ll \varepsilon t, \quad \text{where } \lambda = \frac{1}{4} + t^2. \tag{0.12}$$

Thus

$$m(\lambda) = O(\sqrt{\lambda}).$$

We want to improve the local bound (0.12). Fix $g, h \in C_c^\infty(\mathbb{R})$ and

$$g_{\varepsilon,t}(x) = g(\frac{x}{\varepsilon})e^{tx}, \quad h_{\varepsilon,t}(x) = \frac{1}{\varepsilon}h(\frac{t-x}{\varepsilon}).$$

On the spectral side $\geq \frac{1}{\varepsilon}\sum_{|t_j - t|<\varepsilon} 1$.

What about the geometric side?

identity element: will give t.

hyperbolic contribution: trivial bound

$$\sum_{\substack{P \\ \log P \leq \frac{1}{\varepsilon}}} \frac{g_{\varepsilon,t}(\log P)\log P}{\sqrt{P}}$$

We need

$$\sum_{\substack{P \\ \log P \leq \frac{1}{e}}} \frac{\log P}{\sqrt{P}} \leq t.$$

By PNT for closed geodesics, we have

$$\text{LHS} \gg e^{\frac{1}{2\varepsilon}}.$$

Therefore, we can only take

$$\varepsilon \sim \frac{1}{\log t}.$$

Hence we get

$$*T^2 + *T\log T + *T + O(\frac{T}{\log T}).$$

Main open problem: improve this i.e.

$$m(\lambda) = o\left(\frac{\sqrt{\lambda}}{\log \lambda}\right).$$

References

[1] P. Sarnak, Letter to Zeev Rudnick.