Spectral analysis for $\Gamma\backslash\mathbb{H}$

Erez Lapid

§5 Cuspidal spectrum of $\Gamma\backslash\mathbb{H}$ (February 26, 2009)

Let Γ be the modular group $SL_2(\mathbb{Z})$, and the cusp form is defined by

$$f_P(y) \equiv 0,$$

where

$$f_P(y) = \int_0^1 f(x+iy)\mathrm{d}x.$$

The Laplace operator Δ has discrete spectrum on $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$. In the last time we modified the invariant integral operators to get compact operators on $L^2(\Gamma\backslash\mathbb{H})$, which does not change on cuspidal point-discrete spectrum for Δ.

Lemma. *If f is of moderate growth and is Γ_∞-invariant, $(\Delta+\lambda)f = 0$. Then $f - f_P$ is rapidly decreasing. i.e., $f(z) - f_P(z) \ll (\mathrm{Im}z)^{-N}$ as $\mathrm{Im}z \to \infty$ for any N.*

Proof. In fact, we show that for f in the image of L, i.e., for

$$\begin{aligned} Lf(z) &= \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w) \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} \sum_{n\in\mathbb{Z}} k(z,w+n)f(w)\mathrm{d}\mu(w). \end{aligned}$$

And

$$\begin{aligned} (Lf)_P(z) &= \int_{\Gamma_\infty\backslash\mathbb{H}} \int_o^1 \sum_{n\in\mathbb{Z}} k(z+x,w+n)f(w)\mathrm{d}\mu(w)\mathrm{d}(x) \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} \int_{\mathbb{R}} k(z,w+x)f(w)\mathrm{d}\mu(w)\mathrm{d}(x). \end{aligned}$$

Then we obtain

$$Lf - (Lf)_P(z) = \int_{\Gamma_\infty\backslash\mathbb{H}} \Big(\sum_{n\in\mathbb{Z}} k(z,w+n) - \int_{\mathbb{R}} k(z,w+x)\mathrm{d}x\Big) f(w)\mathrm{d}\mu(w).$$

We showed that

$$\widehat{K}(z,w) \ll (\mathrm{Im}z\mathrm{Im}w)^{-N}$$

for $\mathrm{Im}z \gg 1$, where $\widehat{K}(z,w) = \sum_{n\in\mathbb{Z}} k(z,w+n) - \int_{\mathbb{R}} k(z,w+x)\mathrm{d}x$. We also deduced that $\mathrm{Im}w \gg 1$, if $\widehat{K}(z,w) \neq 0$.

Now, as before,

$$(\mathrm{Im}z)^n(Lf(z) - (Lf)_P(z)) = \int_{\Gamma_\infty\backslash\mathbb{H}} (\mathrm{Im}z)^n \widehat{K}(z,w)(\mathrm{Im}w)^n(\mathrm{Im}w)^{-n} f(w)\mathrm{d}w$$

is bounded for $n \gg 0$, as $(\mathrm{Im}w)^{-n}f(w)$ is bounded.

In the following we want to study the rest of the spectral decomposition of Δ on $L^2(\Gamma\backslash\mathbb{H})$.

Question. How to construct Γ-invariant eigenfunctions of Δ on $\mathbb{H}$?
We start with any eigenfunction f and construct the map

$$z \mapsto \sum_{\gamma\in\Gamma} f(\gamma z).$$

If the summation converges, it will be a Γ-invariant eigenfunction with the same eigenvalue of f.
Important case: $f(z) = (\mathrm{Im} z)^s$, $\lambda = s(1-s)$, then $f(z)$ is N-invariant, and so Γ_∞-invariant.
Def. The Eisenstein series is given by

$$E(z; s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\gamma z)^s$$

where $z \in \mathbb{H}$.

EX. $\Gamma_\infty\backslash\Gamma \mapsto (c, d) \in \mathbb{Z}^2\backslash\{(0, 0)\}$, $\gcd(c, d) = 1$ is well defined, where

$$\Gamma_\infty\backslash\Gamma = \left\{\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbb{Z}\right\} \Big\backslash \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\right\}.$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have $\mathrm{Im}\gamma z = \frac{\mathrm{Im} z}{|cz+d|^2}$.

Thus

$$E(z, s) = \sum_{(m,n)=1} \frac{y^s}{|mz + n|^{2s}} = y^s \sum_{(m,n)=1} \frac{1}{(m^2y^2 + (n + mx)^2)^s}.$$

Properties of the Eisenstein series.

(1) The series converges absolutely and locally uniformly for $\mathrm{Re}s > 1$, even

$$\sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{(m^2y^2 + (n + mx)^2)^s} = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{Q_z(m, n)^s}$$

has the same behavior as $\sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{(m^2+n^2)^s}$, where $Q_z(m, n) = m^2y^2 + (n + mx)^2$ is a quadratic form.

(2) We have $\Delta_z E(z, s) = \lambda E(z, s)$ for $\lambda = s(1 - s)$, and $E(\gamma z, s) = E(z, s)$ for all $\gamma \in \Gamma$, $z \in \mathbb{H}$.

The Fourier expansion of $E(z, s)$ is given by

$$E(z, s) = \sum_{r\in\mathbb{Z}} a_r(z, s)e(rx),$$

where $a_r(z,s) = (E, e(rx))$ and $e(x) = e^{2\pi i x}$. Then

$$\begin{aligned}
a_r(z,s) &= \int_0^1 E(z,s)e(rx)\mathrm{d}x = \int_0^1 \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}} e(rx)\mathrm{d}x \\
&= \int_0^1 y^s e(rx)\mathrm{d}x + y^s \sum_{\substack{(m,n)=1 \\ m\geq 1}} \int_0^1 \frac{e(rx)}{|mz+n|^{2s}}\mathrm{d}x \\
&= \int_0^1 y^s e(rx)\mathrm{d}x + y^s \sum_{m=1}^{\infty} \int_0^1 \frac{1}{m^{2s}} \sum_{\substack{n \\ (m,n)=1}} \frac{e(rx)}{|z+\frac{n}{m}|^{2s}}\mathrm{d}x \\
&= \int_0^1 y^s e(rx)\mathrm{d}x + y^s \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \int_0^1 \sideset{}{^*}\sum_{c(\mathrm{mod} m)} \sum_{k\in\mathbb{Z}} \frac{e(rx)}{|z+\frac{c}{m}+k|^{2s}}\mathrm{d}x \\
&= \int_0^1 y^s e(rx)\mathrm{d}x + y^s \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sideset{}{^*}\sum_{c(\mathrm{mod} m)} \int_{\mathbb{R}} \frac{e(rx)}{|iy+\frac{c}{m}+x|^{2s}}\mathrm{d}x \\
&= \int_0^1 y^s e(rx)\mathrm{d}x + y^s \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sideset{}{^*}\sum_{c(\mathrm{mod} m)} e\Big(\frac{rc}{m}\Big) \int_{\mathbb{R}} \frac{e(rx)}{|iy+x|^{2s}}\mathrm{d}x.
\end{aligned}$$

Firstly,we get the constant term for $r=0$

$$y^s + y^s \sum_m \frac{1}{m^{2s}}\varphi(m) \int_{\mathbb{R}} \frac{\mathrm{d}x}{|iy+x|^{2s}} = y^s + \phi(s)y^{1-s},$$

where

$$\phi(s) = \frac{\xi^*(2s-1)}{\xi^*(2s)}$$

and

$$\xi^*(s) = \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Finally, we get

$$E(z,s) = y^s + \phi(s)y^{1-s} + \sum_r \pi^s \frac{|r|^{s-1}}{\Gamma(s)} \zeta^{-1}(2s) \sum_{d|r} d^{1-2s} W_s(rz),$$

where

$$W_s(rz) = K_{s-1/2}(2\pi n y)e(rx).$$

In particular, we can get meromorphic continuation of $E(z,s)$, and functional equation $E(z,s) = \phi(s)E(z,1-s)$, $\phi(s)\phi(1-s) = 1$. Here we used the analytic properties of the constant term, namely, the Riemann zate function. One can turn the table and prove analytic properties of $\phi(s)$ from these of $E(z,s)$.

Selberg had several proofs of analytic continuation of Eisenstein series. Bernstein had a more conceptual version. This and other useful things can be found in the following web:

www.math.uchicago.edu/∼ mitya/langlands.html

Spectral decomposition.

Just like the Harmonic analysis of $L^2(\mathbb{R}) = \int_{\mathbb{R}} e^{2\pi i x} \mathrm{d}x$, where $e^{2\pi i x}$ is not in $L^2(\mathbb{R})$. One starts with rapidly decreasing functions rather than L^2-functions.

To analyze the continuous spectrum of $L^2(\Gamma\backslash\mathbb{H})$, we will get $E(z;\, s)$ into this picture eventually. We need to study not only continuous families of eigenfunctions (Eisenstein series), but also "wave packets".

We start with $f \in \mathcal{C}_c^\infty\, (\mathbb{R}_{>0})$, define

$$E_f(z) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\gamma z).$$

Notice that $E_f(z)$ is Γ-invariant, but it is not an eigenfunction. For any given z, only finitely many terms in the sum are not zero, because for any given $c > 0$, there are only finitely many γ with $\mathrm{Im}\gamma z > c$.

Moreover, $E_f(z)$ is compactly supported modulo Γ, i.e., $E_f(z) = 0$ if $\mathrm{Im}z \gg 1$.

Since

$$\mathrm{Im}\gamma z \leq \frac{1}{\mathrm{Im}z}, \quad \gamma \notin \Gamma_\infty,$$

Thus, $f(\mathrm{Im}\gamma z) = 0$, $\forall\ \gamma \notin \Gamma_\infty$ and also $f(\mathrm{Im}z) = 0$.

For any given Γ-invariant φ with moderate growth, we have

$$(E_f,\, \varphi)_{\Gamma\backslash\mathbb{H}} = \int_0^\infty f(y)\overline{\varphi_P(y)}\ \frac{\mathrm{d}y}{y^2}. \tag{0.1}$$

Indeed,

$$\begin{aligned}
(E_f,\, \varphi)_{\Gamma\backslash\mathbb{H}} &= \int_{\Gamma\backslash\mathbb{H}} E_f(z)\overline{\varphi(z)}\ \mathrm{d}\mu(z) \\
&= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\gamma z)\overline{\varphi(\gamma z)}\ \mathrm{d}\mu(z) \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} f(\mathrm{Im}z)\overline{\varphi(z)}\ \mathrm{d}\mu(z) \\
&= \int_0^1 \int_0^\infty f(y)\overline{\varphi(x+iy)}\ \frac{\mathrm{d}x\mathrm{d}y}{y^2}. \\
&= \int_0^\infty f(y)\overline{\varphi_P(y)}\ \frac{\mathrm{d}y}{y^2}
\end{aligned}$$

In particular, E_f is orthogonal to all cusp forms (the same thing for $E(z;\, s)$).

Question. What is the relation between these E_f's and the Eisenstein series? (via Mellin inversion).

The Mellin transform is defined as follows:

$$\hat{f}(s) = \int_0^\infty f(x)x^{-s}\frac{\mathrm{d}x}{x}.$$

$\hat{f}(s)$ is entire in s and rapidly decreasing in any vertical strips, i.e., $\forall\, n$, a and b, $(1+|s|)^n\hat{f}(s)$ is bounded on $a \leq \Re s \leq b$.

By Mellin inversion,

$$f(y) = \frac{1}{2\pi i}\int_{\Re s = s_0} \hat{f}(s) y^s \mathrm{d}s$$

for any s_0.

We turn back to E_f, by Mellin Inversion

$$\begin{aligned} E_f(z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\gamma z) \\ &= \frac{1}{2\pi i}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Re s=s_0}\hat{f}(s)(\mathrm{Im}\gamma z)^s\mathrm{d}s. \end{aligned}$$

We can switch sum and integral provided that they converge as a double integral. i.e., for $s_0 > 1$, we have

$$\begin{aligned} E_f(z) &= \frac{1}{2\pi i}\int_{\Re s=s_0}\hat{f}(s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathrm{Im}\gamma z)^s\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\Re s=s_0}\hat{f}(s)E(z;\, s)\mathrm{d}s. \end{aligned}$$

As a matter of fact, we can shift the integral line to $\Re s = 1/2$ later.

What is $(E_f,\, E_g)_{\Gamma\backslash\mathbb{H}}$?

$$\begin{aligned} (E_f,\, E_g)_{\Gamma\backslash\mathbb{H}} &= \left(\frac{1}{2\pi i}\int_{\Re s=s_0}\hat{f}(s)E(z;\, s)\mathrm{d}s,\, E_g\right)_{\Gamma\backslash\mathbb{H}} \\ &= \frac{1}{2\pi i}\int_{\Re s=s_0}\hat{f}(s)\big(E(z;\, s),\, E_g\big)_{\Gamma\backslash\mathbb{H}}\mathrm{d}s, \end{aligned}$$

where the inner product is

$$\begin{aligned} (E(z;\, s),\, E_g)_{\Gamma\backslash\mathbb{H}} &= \int_0^\infty E_p(y,\, s)\overline{g(y)}\,\frac{\mathrm{d}y}{y^2} \\ &= \int_0^\infty (y^s + \phi(s)y^{1-s})\overline{g(y)}\,\frac{\mathrm{d}y}{y^2} \\ &= \bar{\hat{g}}(1-s) + \phi(s)\bar{\hat{g}}(s). \end{aligned}$$

Therefore,

$$(E_f,\, E_g)_{\Gamma\backslash\mathbb{H}} = \frac{1}{2\pi i}\int_{\Re s=s_0}\hat{f}(s)\left(\bar{\hat{g}}(1-s)+\phi(s)\bar{\hat{g}}(s)\right)\mathrm{d}s.$$

Remark.

We just want to get something like

$$(f,\, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\hat{f}(t)\,\overline{\hat{g}(t)}\,\mathrm{d}t.$$

Another version of E_f : truncation. $E(z;\, s)$ is of moderate growth, but not in L^2 for any s. It is "almostly" on L^2 for $\Re s = 1/2$.

Def. Let T be a fixed parameter, the truncation operator is defined by

$$\Lambda^T E(z;\, s) = E^T(z;\, s) = \begin{cases} E(z;\, s), & \mathrm{Im} z \leq T, \\ E(z;\, s) - E_P(y), & \mathrm{Im} z > T, \end{cases}$$

where $z \in \mathcal{F}$. Obviously, $E^T(z;\, s)$ is rapidly decreasing.

Maass-Selberg relations.

$$\left(\Lambda^T E(\cdot, s_1),\, \overline{\Lambda^T E(\cdot, s_2)}\right)_{\mathcal{F}} = \frac{T^{s_1 - s_2 - 1}}{s_1 - s_2 - 1} + \frac{\phi(s_1) T^{s_2 - s_1}}{s_2 - s_1} + \frac{\phi(s_2) T^{s_1 - s_2}}{s_1 - s_2} + \frac{\phi(s_1)\phi(s_2)}{1 - s_1 - s_2} T^{1 - s_1 - s_2}.$$

From the Maass-Selberg relation, we have the following consequences:
(1) $E(z;\, s)$ and $\phi(s)$ have the same poles;
(2) They are holomorphic for $\Re s = 1/2$;
(3) There are only finitely many poles for $\Re s = 1/2$, they are all simple and attained for s real.

Proof of part 3.

We fix $\sigma > 1/2$, $s_0 = \sigma + i\tau$ and want to take $s_2 = \bar{s_1}$, $s_1 = s_0 + it$. Here t is small. Applying the Maass-Selberg relation, we obtain

$$\| E^T(\cdot,\, s_1) \|^2 = \frac{T^{2\sigma - 1}}{2\sigma - 1} + \mathrm{Im} \frac{\overline{\phi(s_1)} T^{2i(t+\tau)}}{2i(t+\tau)} + \frac{T^{1-2\sigma}}{1 - 2\sigma} |\phi(s_1)|^2. \tag{0.2}$$

Then we get

$$\frac{T^{2\sigma - 1}}{2\sigma - 1} + \mathrm{Im} \frac{\overline{\phi(s_1)} T^{2i(t+\tau)}}{2i(t+\tau)} \geq \frac{T^{1-2\sigma}}{2\sigma - 1} |\phi(s_1)|^2,$$

since the left hand side of (0.2) ≥ 0.

Suppose that $\phi(s_1)$ has a pole of order $m \geq 1$ at s_0. Multiplying t^{2m} on both sides, we have

$$\frac{t^{2m} T^{2\sigma - 1}}{2\sigma - 1} + t^{2m} \mathrm{Im} \frac{\overline{\phi(s_1)} T^{2i(t+\tau)}}{2i(t+\tau)} \geq \frac{T^{1-2\sigma}}{2\sigma - 1} t^{2m} |\phi(s_1)|^2. \tag{0.3}$$

The first term of the left hand side of (0.3) tends to 0. The second term of the left hand side of (0.3) tends to 0 unless $m = 1$ and $\tau = 0$. The right hand side of (0.3) > 0, as t tends to 0.