Spectral analysis for $\Gamma\backslash\mathbb{H}$

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Let Γ be a discrete group of $SL_2(\mathbb{R})$, then Γ acts on $\mathbb{H}$ discontinuously. We are interested in the case that $\mathrm{Vol}(\Gamma\backslash\mathbb{H}) < \infty$, especially the case $\Gamma = SL_2(\mathbb{Z})$, since this case includes many (but not all) features of the general case.

Fundamental Domain.

A Fundamental domain for Γ is a domain $\mathcal{F}$ in $\mathbb{H}$ which satisfies: i) $\gamma\mathcal{F}\bigcap\mathcal{F} = \emptyset$, for any $\gamma(\neq 1) \in \Gamma$; ii) $\forall x \in \mathbb{H}$, $\Gamma x \bigcap \overline{\mathcal{F}} \neq \emptyset$.

A fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ is given by

$$\mathcal{F} = \{z \in \mathbb{H} : -\frac{1}{2} < \mathrm{Re}z < \frac{1}{2}, |z| > 1\}.$$

Automorphic Forms.

Automorphic forms are functions on $\mathbb{H}$ which satisfy the following conditions.

Firstly, it is a function on $\Gamma\backslash\mathbb{H}$, i.e., certain Γ-invariant functions on $\mathbb{H}$. For example, the j-function

$$j : \Gamma\backslash\mathbb{H} \longrightarrow \mathbb{C}.$$

Secondly, it is an eigenfunction of Δ. The hyperbolic metric $\frac{|\mathrm{d}z|}{\mathrm{Im}z}$ can descend to $\Gamma\backslash\mathbb{H}$, since it is $G-$invariant. Similarly, Δ descends to $\Gamma\backslash\mathbb{H}$, and it becomes Laplacion on $\Gamma\backslash\mathbb{H}$.

Thirdly, it is of moderate growth. A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called of moderate growth, if there exists n s.t. $f(z) \ll (\mathrm{Im}z)^n$. Without this restriction, there are many kinds of eigenfunctions of Δ. For example, the harmonic functions $\mathrm{Re}\, j$, e^j, $e^{e^j} \cdots$ are all eigenfunctions of Δ with eigenvalue 0.

We restrict to the space of automorphic forms with eigenvalue λ.

Theorem 1. *The space of Γ-invariant functions with eigenvalue λ which are of moderate growth is finite-dimensional.*

To prove the theorem, the main object is the constant term in the Fourier expansion of such functions.

Suppose $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a Γ_∞-invariant function, i.e., $f(z+1) = f(z)$. We define

$$f_P(y) = \int_0^1 f(x+iy)\mathrm{d}x = \int_{\Gamma_\infty\backslash N} f(n\cdot iy)\mathrm{d}n,$$

where

$$\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbb{Z}\right\},$$

$$N = \left\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R}\right\}.$$

In fact, we have exact sequence

$$C_{\text{cusp}}(\Gamma\backslash\mathbb{H})_\lambda \longrightarrow C(\Gamma\backslash\mathbb{H})_\lambda \longrightarrow C(\mathbb{R}_{>0})_\lambda.$$

Namely, the map $f \mapsto f_P$ gives a homomorphism from the space of automorphic forms with eigenvalue λ to the space of $N-$invariant eigenfunctions, and functions in kernel are called cusp forms. For given $\lambda = s(1-s)$, the image is spanned by y^s and y^{1-s} if $s \neq 1/2$, or by $\sqrt{y}$ and $\sqrt{y}\log y$ if $s = 1/2$. Therefore, Theorem 1 reduces to show the finite-dimensionality of the space of cuspidal forms with eigenvalues λ of moderate growth.

Recall invariant integral operator,

$$Lf = \int_{\mathbb{H}} k(z,w)f(w)d\mu(w),$$

where $k(z,w)$ is a point-pair invariant function. We know

$$C(\mathbb{H}) \xrightarrow{L} C(\mathbb{H}),$$

then $L \circ T_g = T_g \circ L \Rightarrow L(C(\Gamma\backslash\mathbb{H})) \subseteq C(\Gamma\backslash\mathbb{H})$, i.e.,

$$C(\Gamma\backslash\mathbb{H}) \xrightarrow{L} C(\Gamma\backslash\mathbb{H}).$$

We want to view L as an integral operator on $\Gamma\backslash\mathbb{H}$. By separating variables, we have

$$\begin{aligned} Lf(z) &= \int_{\mathbb{H}} k(z,w)f(w)\mathrm{d}\mu(w) \\ &= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma} k(z,\gamma w)f(w)\mathrm{d}\mu(w) \\ &= \int_{\Gamma\backslash\mathbb{H}} K(z,w)f(w)\mathrm{d}\mu(w), \end{aligned}$$

where the new kernel

$$K(z,w) \overset{(*)}{=} \sum_{\gamma\in\Gamma} k(z,\gamma w)$$

is a function on $\Gamma\backslash\mathbb{H} \times \Gamma\backslash\mathbb{H}$ called the automorphic kernel.

Remark 1. If $k \in C_c^\infty(\mathbb{R}_+)$, then there are only finitely many summands in (*) which are non-zero. It shows that $K(z,w)$ is well defined.

Remark 2. K is not bounded. If $z = w$ is high up in the cusp, many $\gamma' s$ of the form $\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right)$ will contribute(up to $n \ll \mathrm{Im} z$). However, other $\gamma' s$ do not cause a problem. More specially, we have

Lemma. *$\exists C \subset \mathcal{F}$ compact s.t. $z,w \in \mathcal{F}$, $k(\gamma z,w) \neq 0 \Rightarrow \gamma \in \Gamma_\infty$ or $z,w \in C$.*

Proof. We know $k(\gamma z,w) \neq 0$ implies that $u(\gamma z,w)$ is bounded, since $k \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. It is easy to check

$$\mathrm{Im}\gamma z \cdot \mathrm{Im} z \leq 1, \quad \text{if} \gamma \notin \Gamma_\infty.$$

Assume $\mathrm{Im} z \gg 1$ and $\gamma \notin \Gamma_\infty$, we have

$$\mathrm{Im} w \geq \frac{\sqrt{3}}{2} \Rightarrow u(\gamma z, w) \gg 0 \Rightarrow k(\gamma z, w) = 0.$$

Similar result for the case $\mathrm{Im} w \gg 1$, $\mathrm{Im}\gamma z \leq 1$ where $\gamma \notin \Gamma_\infty$, since

$$\Gamma_\infty \cdot \mathcal{F} \supseteq \{z \in \mathbb{H} : \mathrm{Im} z \geq 1\} \Rightarrow \{z \in \mathcal{F}, \gamma \notin \Gamma_\infty \Rightarrow \mathrm{Im}\gamma z \leq 1\}.$$

So again, $u(\gamma z, w) \gg 0$.

Thus we can take $C = \{z \in \mathcal{F}, \mathrm{Im} z \leq T\}$ for T sufficiently large. □

Recall a cusp form f is a Γ-invariant of moderate growth function which satisfies

$$f_P = 0 \text{ and } (\Delta + \lambda) f = 0.$$

We know f is an eigenfunction of all invariant integral operators, i.e., $Lf = \Lambda f$, where Λ depends only on λ but not on f. Viewed on $\Gamma\backslash\mathbb{H}$, L is not compact, since $K(z, w)$ is unbounded and $K \notin L^2(\Gamma\backslash\mathbb{H} \times \Gamma\backslash\mathbb{H})$. However, we can modify K as follows.

Define

$$H(z, w) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathbb{R}} k(z, n(x)\gamma w) dx$$

which is Γ-invariant in w. Let $\widehat{K}(z, w) = K(z, w) - H(z, w)$, then

$$\widehat{L} f(z) = \int_{\Gamma\backslash\mathbb{H}} \widehat{K}(z, w) f(w) d\mu(w), \quad z \in \mathcal{F}.$$

We know $\widehat{L}$ is a map from $C(\Gamma\backslash\mathbb{H})$ to $C(\mathcal{F})$.

Lemma 1. *If $f_P = 0$, then $\widehat{L} f = Lf$.*

Proof.

$$\begin{aligned}
(L - \widehat{L}) f &= \int_{\Gamma\backslash\mathbb{H}} H(z, w) f(w) d\mu(w) \\
&= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\mathbb{R}} k(z, n(x)\gamma w) f(w) \mathrm{d}x \mathrm{d}\mu(w) \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} \int_{\mathbb{R}} k(z, n(x) w) f(w) \mathrm{d}x \mathrm{d}\mu(w) \\
&\overset{w=u+iv}{=\!=\!=} \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}} k(z, u + x + iv) f(u + iv) \mathrm{d}x \frac{\mathrm{d}v \mathrm{d}u}{v^2} \\
&\overset{x \mapsto x-u}{=\!=\!=} \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}} k(z, x + iv) f(u + iv) \mathrm{d}x \frac{\mathrm{d}v \mathrm{d}u}{v^2} \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \Big(\int_0^1 f(u + iv) \mathrm{d}u \Big) k(z, x + iv) \mathrm{d}x \frac{\mathrm{d}v}{v^2} \\
&= 0 \text{ (by cuspidality).} \quad \square
\end{aligned}$$

Lemma 2. *$\widehat{K}(z,w)$ is rapidly decreasing on $\mathcal{F}\times\mathcal{F}$. i.e., for any n, $\widehat{K}(z,w)\le(\mathrm{Im}z\mathrm{Im}w)^n$. In particular, $\widehat{K}(z,w)$ is bounded $\Rightarrow \widehat{K}(z,w)\in L^2(\mathcal{F}^2)$.*

Corollary. *$L\mid_{L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})}$ is compact.*

Proof of the Corollary. By Lemma 1, we have

$$L\mid_{L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})}=\widehat{L}.$$

$\widehat{L}$ is an integral operator on F with kernel $\widehat{K}\in L^2(\mathcal{F}^2)$. This implies that $L\mid_{L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})}$ is compact. $\square$

Proof of Lemma 2 . We come back to give the proof of Lemma 2. From previous lemma, there exists a compact subset C of $\mathcal{F}$ such that

$$K(z,w)=\sum_{\gamma\in\Gamma_\infty}k(z,\gamma w)\quad\text{unless } z,w\in C.$$

Similarly, the same argument shows that

$$H(z,w)=\int_{\mathbb{R}}k(z,n(x)w)\mathrm{d}x\quad\text{unless } z,w\in C.$$

We only need to consider the difference

$$\sum_{\gamma\in\Gamma_\infty}k(z,\gamma w)-\int_{\mathbb{R}}k(z,w+x)\mathrm{d}x. \tag{0.1}$$

Recall Poisson summation formula

$$\sum_{n\in\mathbb{Z}}g(n)=\sum_{n\in\mathbb{Z}}\hat{g}(n)$$

which implies

$$\sum_{n\in\mathbb{Z}}g(n)-\hat{g}(0)=\sum_{n\neq0}\hat{g}(n),$$

where $\hat{g}(n)=n^{-k}$, $g_n^{(k)}\le n^{-k}||g^{(k)}||_1$. So the difference (0.1) is bounded in terms of the derivatives in the x variable of $k(z,w+x)$. Denote $z=x'+iy'=n(x')a(y')i$, then we have

$$\begin{aligned}k(z,w+x)&=k(n(x')a(y')i,n(x)w)=k(n(-x)n(x')a(y')i,w)\\&=k(n(x'-x)a(y')i,w)=k(a(y')n(y'^{-1}(x'-x))i,w)\\&=k(n(y'^{-1}(x'-x))i,w').\end{aligned}$$

The n-th derivative $\le(y')^{-n}*$ derivative of k. Similarly, it is bounded by $(\mathrm{Im}w)^{-n}$. $\square$

By properties of compact operator, we have

Corollary 1. *The space of cusp forms of eigenvalues λ is finite-dimensional.*

Corollary 2. *Every cusp form (moderate growth and eigenfunction of Δ) is rapidly decreasing on F.*

Corollary 3. *Δ has discrete spectrum on $L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})$.*

Firstly, let us prove Corollary 2. Suppose f is a cusp form of moderate growth and is Γ-invariant, $(\Delta+\lambda)f=0$. Then there exists k such that

$$L_k f = \Lambda f, \quad \Lambda \neq 0.$$

So

$$\widehat{L}_k f = \Lambda f = \int_{\mathcal{F}} \widehat{K}(z,w) f(w) \mathrm{d}\mu w. \tag{0.2}$$

Lemma 2 told us that $\widehat{K} f(z)$ is rapidly decreasing because

$$(\mathrm{Im} z)^n \widehat{L} f(z) = \int (\mathrm{Im} z)^n \widehat{K}(z,w) (\mathrm{Im} w)^n \frac{f(w)}{(\mathrm{Im} w)^n} \mathrm{d}\mu w$$

where $n \gg 0$. So the left hand side is bounded, and then $\widehat{L}_k f$ is rapidly decreasing.

Corollary 1 follows from compactness of $\widehat{L}_k$ and (0.2). To show Corollary 3, L_k has discrete spectrum on $L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})$. Let $U =$ span eigenfunctions of L_k for all k.

Claim: $U =$ all cuspidal spectrum. Otherwise, for any $\varphi(\neq 0) \in L^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})$,

$$L_k \varphi = 0 \quad \forall k.$$

It is easy to show that it is impossible.