EXERCISE

1. Let k be the function whose graph is

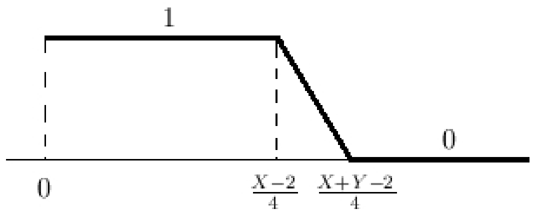

FIGURE 1

where $2 \le Y \le X$, $X = e^A + e^{-A}$, $X + Y = e^B + e^{-B}$. Let

$$q(v) = \int_v^\infty \frac{h(u)}{\sqrt{u-v}} du,$$

$$g(r) = 2q\left(\frac{e^r + e^{-r} - 2}{4}\right),$$

$$h(t) = \int_{-\infty}^{\infty} e^{irt} g(r) dr.$$

a) We have

$$\frac{2}{3} q(v) = \frac{1}{Y}\Big[\max(0, \frac{X+Y-2}{4} - v)^{3/2} - \max(0, \frac{X-2}{4} - v)^{3/2}\Big].$$

b)

$$h(t) \ll \log X \sqrt{X}.$$

c) For $t \in \mathbb{R}$, show that

$$|h(t)| \le \frac{1}{t^2} \int_0^\alpha |g''(r)| dr + \frac{1}{t} \int_\alpha^A |g'(r)| dr + g(A) \ll \frac{X}{t^2 \sqrt{a}} + \frac{\sqrt{a}}{t} + \frac{\sqrt{Y}}{t},$$

for any $0 \le \alpha \le A$, where $e^\alpha + e^{-\alpha} = X - a$. Deduce that

$$h(t) \ll \frac{\sqrt{X}}{t^{3/2}}$$

for $1 \le t \le \frac{X}{Y}$.

d) Show that

$$\begin{aligned} |h(t) &\le \frac{1}{t^3}\Big(\int_0^\alpha |g'''(r)| dr + \int_A^\beta |g'''(r)| dr\Big)| + \frac{1}{t^2}\Big(\int_\alpha^A |g''(r)| dr + \int_\beta^B |g''(r)| dr\Big) \\ &\ll \frac{1}{t^3} \frac{X^2}{\sqrt{a} Y} + \frac{1}{t^2} \frac{X\sqrt{a}}{Y} + \frac{1}{t^3} \frac{X^2}{\sqrt{b} Y} + \frac{1}{t^2} \frac{X\sqrt{b}}{Y} \end{aligned}$$

for any $0 \leq \alpha \leq A$, $A \leq \beta \leq B$, where $e^{\alpha} + e^{-\alpha} = X - a$, $e^{\beta} + e^{-\beta} = X + Y - b$. Deduce that

$$|h(t)| \ll \frac{1}{t^{5/2}} \frac{X}{Y} \sqrt{X}$$

for $\frac{X}{Y} \leq t$.

e) By approximating $g(r)$ by $q(\frac{e^r - 2}{4})$ show that for $\frac{1}{2} < s \leq 1$,

$$h(t) = 2B(s - \frac{1}{2}, 3/2) X^s + O(Y + \sqrt{X})$$

where B is the Beta function (The implied constant depends on s).

f) Finally, deduce that

$$P(X) = \sum_{\frac{2}{3} < s_j \leq 1} 2B(s_j - \frac{1}{2}, \frac{3}{2}) u_j(z) \overline{u_j(w)} X^{s_j} + O(X^{2/3})$$

where $P(X) := \sharp\{\gamma \in \Gamma : 4u(\gamma z, w) + 2 \leq X\}$, and u_j is an orthonormal basis of eigenfunctions

$$\Big(\Delta + s_j(1 - s_j)\Big) u_j = 0.$$