Langlands picture of automorphic forms and L-functions

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§7 Automorphic forms on Adele group (Mar. 20)

We know both $W_s(\Gamma)$ and $S_k(\Gamma)$ can be embedded in $\mathcal{L}_0^2(\Gamma\backslash G)$. Denote $\mathcal{L}_d^2(\Gamma\backslash G)$ as the discrete part of the spectrum. We have

Theorem 7.1. *$\mathcal{L}_d^2(\Gamma\backslash G)$ is the direct sum of all irreducible unitary representations of G that occur in $\mathcal{L}^2(\Gamma\backslash G)$, and*
*(1)***(Gelfand, Graev and Piatetski-Shapiro)** *$\mathcal{L}_d^2 = \mathcal{L}_0^2 \oplus 1$ and the spectrum of $\mathcal{L}_0^2$ is finite. (It means that the multiplicity of irreducible unitary representation occurring in $\mathcal{L}_0^2$ is finite.)*
*(2)***(Selberg, 1950's)** *The continuous spectrum of $\mathcal{L}^2$ is denoted by $\mathcal{L}_{cont}^2$, so that*

$$\mathcal{L}_{cont}^2 = \oint \pi^{it}\, dt,$$

where π^{it} is the principal series.

Remark 1. In fact, the multiplicity of each irreducible unitary representation occurring in $\mathcal{L}_0^2(\Gamma\backslash G)$ is one. This was proved by Jacquet and Langlands. In 1950's, Selberg finished everything about the spectrum by working on the upper half plane $\mathbb{H}$. In 1965, Langlands did this for an arbitrary reductive group, and the whole Langlands program was proposed in 1967.

Now we succeeded in lifting the classical automorphic form up to automorphic form of $SL_2(\mathbb{R})$. The next question is about the Hecke operators. In fact, there is no group theory to handle this only on $SL_2(\mathbb{R})$. The ideal is to lift it to functions of $GL_2(\mathbb{A})$. Via the method of GL_1 theory appearing in Tate's thesis, Jacquet and Langlands succeeded in doing it for $GL_2(\mathbb{A})$.

In what follows we will work on $\mathbb{Q}$. Let $G = GL_2$, $G_p = GL_2(\mathbb{Q}_p)$ with $p \le \infty$, and let

$$K_p = \begin{cases} \mathcal{O}(2,\mathbb{R}) & \text{if } p = \infty, \\ GL(2,\mathbb{Z}_p) & \text{if } p < \infty. \end{cases}$$

Define

$$\begin{aligned} G_{\mathbb{A}} &= {\prod_{p\le\infty}}' G_p \\ &= \left\{(g_\infty, g_2, g_3, g_5, \ldots),\text{ where } g_p \in K_p \text{ for all but finite many } p\right\}. \end{aligned}$$

Moreover, Z denotes the center of G, and

$$Z_{\mathbb{A}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, z \in \mathbb{A}^\times \right\}.$$

We know the automorphic forms on GL_1 are Grossencharacters, i.e. the functions ψ on $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Note that $\mathbb{A}^\times = GL_1(\mathbb{A})$ and $\mathbb{Q}^\times = GL_1(\mathbb{Q})$. To generalize the GL_1 theory, we need to consider the functions on $G_{\mathbb{Q}}\backslash G_{\mathbb{A}}$. However, it is not compact. The ideal is to consider functions on $G_{\mathbb{Q}}Z_{\mathbb{A}}\backslash G_{\mathbb{A}}$ associated with a character of $Z_{\mathbb{A}}$. Since $Z_{\mathbb{A}} \cong \mathbb{A}^\times$, the center character is just a Grossencharacter.

Now we recall the definition of classical modular forms.

Let

$$\Gamma = \Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \middle| c \equiv 0 (\mathrm{mod} N) \right\}$$

and let ψ be a character of $N\mathbb{Z}\backslash\mathbb{Z}$. Then $S_k(\Gamma_0(N), \psi)$ consists of f which are holomorphic on $\mathbb{H}$, vanish at ever cusp of $\Gamma_0(N)$ and satisfy

$$f\left(\frac{az+b}{cz+d}\right) = \psi(a)(cz+d)^k f(z).$$

In order to lift $f \in S_k(\Gamma_0(N), \psi)$ to a "nice" function on $G_{\mathbb{A}}$, we lift it to $GL_2^+(\mathbb{R})$ firstly, where

$$GL_2^+(\mathbb{R}) = \{g \in GL_2(\mathbb{R}), \det g > 0\}.$$

Proposition 7.2. *For $f \in S_k(\Gamma_0(N), \psi)$ and $g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, denote $j(g_\infty, i) = (cz+d)(\det g_\infty)^{-\frac{1}{2}}$. The function*

$$\varphi_f(g_\infty) = f(g_\infty i) j(g_\infty, i)^{-k}$$

is a "nice" function on $GL_2^+(\mathbb{R})$.

Remark 2. Since $SL_2(\mathbb{R}) = GL_2^+(\mathbb{R})/Z^+(\mathbb{R})$, φ_f is "nice" as in theorem 5.2.

We know the class number of Q equals 1. The strong approximation implies that

$$\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}_+ \times \prod_{p<\infty} \mathbb{Z}_p^\times.$$

And in GL_2, we have

$$G_{\mathbb{A}} = G_{\mathbb{Q}} \times GL_2^+(\mathbb{R}) \times \prod_{p<\infty} K'_p, \tag{7.1}$$

where K'_p is any choice of open subgroup of K_p satisfying $K'_p = K_p$ for almost all p, and $\det(K'_p) = \mathbb{Z}_p^\times$ for every p. Therefore, for $N \in \mathbb{Z}$, one can choose

$$K'_p = K_p^N = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{Z}_p) \middle| c \equiv 0 (\mathrm{mod} N) \right\}.$$

Denote $K_0(N) = \prod_{p<\infty} K_p^N$, we have

$$\Gamma_0(N)\backslash SL_2(\mathbb{R}) \cong Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K_0(N) \tag{7.2}$$

In fact, (7.2) follows form the following proposition.

Proposition 7.3. *We have*

$$G_{\mathbb{Q}} \bigcap \left(GL_2^+(\mathbb{R}) \prod_{p<\infty} K_p^N \right) = \Gamma_0(N). \tag{7.3}$$

Proof. It is easy to check that L.H.S $\supseteq$ R.H.S. Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ L.H.S., then $a, b, c, d \in \mathbb{Q}$, $\det g > 0$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bigcap_{p<\infty} K_p^N.$$

It implies $a, b, c, d \in \bigcap_{p<\infty} \mathbb{Z}_p = \mathbb{Z}$, $\det g \in \bigcap_{p<\infty} \mathbb{Z}_p^\times = \mathbb{Z}^\times$, and $c \equiv 0(\mathrm{mod}N)$. Thus $g \in$ R.H.S. $\square$

(7.3) implies we can view a character ψ of $\mathbb{Z}/N\mathbb{Z}$ as a Grossencharacter, because

$$\mathbb{Z}/N\mathbb{Z} \cong \prod_{p^r \| N} \mathbb{Z}/p^r\mathbb{Z} \cong \prod_{p^r \| N} \mathbb{Z}_p/p^r\mathbb{Z}_p.$$

Via (7.1), (7.2) and (7.3), we can lift $f \in S_k(\Gamma_0(N), \psi)$ to $G_{\mathbb{A}}$ now.

Definition 7.4. With respect to (7.1), we have $g = \gamma g_\infty k$ where $\gamma \in \mathbb{Q}$, $g_\infty \in GL_2^+(\mathbb{R})$ and $k \in K_0(N)$. For any $f \in S_k(\Gamma_0(N), \psi)$, we define a function φ on G_A as

$$\varphi(g) = \varphi(\gamma g_\infty^+ k) = \varphi_f(g_\infty)\psi(k), \tag{7.4}$$

where ψ is a character of $N\mathbb{Z}\backslash\mathbb{Z}$.

It is easy to check that the definition is well defined. In deed, we have

Proposition 7.5. *The map $f \mapsto \varphi$ on $G_{\mathbb{A}}$ is an isomorphism between $S_k(\Gamma_0(N), \psi)$ and the space of functions on $G_{\mathbb{A}}$ such that*
(1) $\varphi(\gamma g) = \varphi(g)$, $\gamma \in G_{\mathbb{Q}}$;
(2) $\varphi(zg) = \varphi(gz) = \psi(z)\varphi(g)$, $z \in Z_{\mathbb{A}}$;
(3) $\varphi(g\kappa(\theta)) = e^{-ik\theta}\varphi(g)$, $\kappa(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$;
(4) $\Delta\varphi(g_\infty) = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\varphi(g_\infty)$;
(5) $\varphi(gk_0) = \varphi(g)\psi(k_0)$, $\forall k_0 \in K_0(N)$;
(6) φ is cuspidal, i.e.

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0, \qquad \textit{for almost all } g.$$

A general definition of automorphic form is given as follows:

Definition 7.6. An automorphic form is a "nice" function φ on $G_{\mathbb{A}}$ such that
(1) $\varphi(\gamma g) = \varphi(g)$, $\gamma \in G_{\mathbb{Q}}$;
(2) $\varphi(zg) = \varphi(gz) = \psi(z)\varphi(g)$, where $z \in Z_{\mathbb{A}}$ and ψ is a Grossencharacter.
(3) φ is right K-finite, where $K = K_\infty \prod_{p<\infty} K_p^N$;
(4) As a function on G_∞, φ is "smooth" and $\mathfrak{z}$-finite, where $\mathfrak{z}$ is the center of the universal enveloping algebra of G_∞.

Next we consider the Fourier expansions for φ. For simplicity, we only consider the case $\Gamma = SL_2(\mathbb{Z})$ and $\psi = 1$. Recall for $f \in S_k(\Gamma)$,

$$f(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n z).$$

Let φ be the corresponding functions in $\mathcal{L}_0^2$. The Fourier expansion of φ is defined as

$$\varphi\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) g\right) = \sum_{\xi \in \mathbb{Q}} \varphi_\xi(g)\tau(\xi x) \quad \text{for almost all } g \in G_{\mathbb{A}},$$

where $\tau(x) = e^{2\pi i x_\infty} \prod_{p<\infty} \tau_p(x_p)$ is a non-trivial additive character of $\mathbb{A}$, τ_p are unramified additive characters of $\mathbb{Q}_p$, and

$$\varphi_\xi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) g\right) \overline{\tau(\xi x)} \mathrm{d}x.$$

Proposition 7.7. *Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$, $\Gamma = SL_2(\mathbb{Z})$ and $\psi = 1$. For the corresponding φ and each $y > 0$, we have*

$$\varphi_\xi\left(\left(\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right)\right) = \begin{cases} a_n e^{-2\pi n y} & \text{if } \xi = n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$\varphi\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right)\right) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} = f(z), \quad z = x + iy.$$

Proof. We know $\mathbb{A} = \mathbb{Q}\mathbb{A}_{S_\infty}$, where $\mathbb{A}_{S_\infty} = \mathbb{R}\prod_{p<\infty} \mathbb{Z}_p$. Therefore,

$$\mathbb{Q}\backslash\mathbb{A} = \left(\mathbb{Q}\bigcap\mathbb{A}_{S_\infty}\right)\backslash\mathbb{A}_{S_\infty} = \mathbb{Z}\backslash\mathbb{R} \times \prod_{p<\infty} \mathbb{Z}_p.$$

Since $\Gamma = SL_2(\mathbb{Z})$ and $\psi = 1$, it implies $N = 1$ and φ is right invariant on $\prod_{p<\infty} K_p$. Assume $\xi = n \in \mathbb{Z}$. By (7.4), we have

$$\begin{aligned} \varphi_n\left(\left(\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right)\right) &= \int_{\mathbb{Z}\backslash\mathbb{R}} \varphi\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right)\right) \overline{\tau_\infty(nx)} \mathrm{d}x \\ &= \int_0^1 f(z) e^{-2\pi i n x} \mathrm{d}x \\ &= a_n e^{-2\pi i n y}. \end{aligned}$$

Now assume $\xi \notin \mathbb{Z}$. There exists some p and some $m > 0$ such that $\xi = \alpha p^{-m}$ with p and α relatively prime. Since φ is right invariant by $\prod_{p<\infty} K_p$ and $\left(\begin{array}{cc} 1 & p^{m-1} \\ 0 & 1 \end{array}\right) \in \prod_{p<\infty} K_p$, we compute

$$
\begin{aligned}
& \varphi_\xi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \\
= & \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p^{m-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -p^{m-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p^{m-1} \\ 0 & 1 \end{pmatrix}\right)\overline{\tau(\xi x)}\mathrm{d}x \\
= & \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x+p^{m-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\overline{\tau(\xi x)}\mathrm{d}x \\
= & \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\overline{\tau\left(\xi(x-p^{m-1})\right)}\mathrm{d}x \\
= & \tau(\xi p^{m-1})\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\mathrm{d}x \\
= & \tau(\alpha p^{-1})\varphi_\xi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}
$$

This implies that $\varphi_\xi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$, since $\tau(\alpha p^{-1}) = \tau_p(\alpha p^{-1}) \neq 1$. $\square$