Final Exam of Graduate Analysis

School of Mathematics, Shandong University

Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two hours to complete this test. Good luck!*

—— 14:00-16:00, Jan 8, 2018.

注意事项：卷面分5分,试题总分95分. 其中卷面整洁,书写规范(5分);卷面较整洁,书写较规范(3分); 书写潦草，乱涂乱画(0分).

1. **(20 points)** Let B be the subgroup of $GL_2(\mathbb{R})$ defined by

$$B = \left\{ \left(\begin{smallmatrix} 1 & x \\ & y \end{smallmatrix} \right) : \ x, y \in \mathbb{R}, y \neq 0 \right\}.$$

(1). Show that $I(f) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} f \left(\begin{smallmatrix} 1 & x \\ & y \end{smallmatrix} \right) \frac{dxdy}{y}$ is a left Haar integral on B

(2). Find the modular function of B.

2. **(25 points)** Let $C[0,1]$ be the normed linear space with the norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$ for any $x = x(t) \in C[0,1]$. Suppose that T, S are operators

$$T : \ x(t) \longmapsto tx(t),$$

$$S : \ x(t) \longmapsto t \int_0^1 x(t) dt.$$

(1), Find the norm of operators $\|T\|$, $\|S\|$, $\|TS\|$ and $\|ST\|$.

(2), Find the spectrums $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ of the operator T.

3. **(25 points)** Let E be a Hilbert space and $T \in B(E)$.

(1). Show that $\|T\| = \sup\{\langle Tx, y \rangle : \ \|x\|, \|y\| \leq 1\}$.

(2). If T is normal, then $\|T\|_\sigma = \|T\|$, where $\|T\|_\sigma$ is the spectral radius of T.

4. **(25 points)** Prove that

(1) if $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$, then $\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|^p dx = 0$.

(2) if $f \in L^1(\mathbb{R}^n)$ and $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot y} dx$, then $\lim_{|y| \to \infty} \widehat{f}(y) = 0$