Spectral analysis for $\Gamma\backslash\mathbb{H}$

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§10 Spectral decomposition, hyperbolic lattice point problem(March 12, 2009)

Recall Bessel's inequality

$$\sum_j |(u, e_j)|^2 \le \|u\|^2,$$

$\mathcal{H}$ is a Hilbert space, and $\{e_j\}$ is the orthogonal system. In our case, we consider functions in $L^2(\Gamma\backslash\mathbb{H})$, $\{e_j\}$ are orthogonal eigenfunctions of Δ.

We take $k = k(u(z,w)) \in C_c^\infty(\mathbb{R} > 0)$, $k \mapsto h$ is the Selberg transform. We define

$$K(z,\ w) = \sum_{\gamma\in\Gamma} k(z,\ \gamma w).$$

Fix w and take $u = (z \mapsto K(z,w))$. If $(\Delta + \lambda_j)\varphi_j = 0$, where $\lambda_j = s_j(1-s_j)$, $s_j = \frac{1}{2} + it_j$, then

$$(u,\ \varphi_j) = (K(\cdot, w),\ \varphi_j) = \Big(k(u(\cdot, w),\ \varphi_j\Big)_{\mathbb{H}} = h(t_j)\varphi_j(w).$$

Also,

$$\Big|(u, \int E(w;\ \frac{1}{2} + ir)\mathrm{d}r)\Big|^2 = \int \Big|h(r)E(w;\ \frac{1}{2}+ir)\Big|^2\mathrm{d}r.$$

The Bessel's inequality implies

$$\sum_j |h(t_j)|^2|\varphi_j(w)|^2 + \int_A^B |h(r)|^2|E(w;\ \frac{1}{2}+ir)|^2\mathrm{d}r \quad\le\quad \|K(\cdot, w)\|^2_{\Gamma\backslash\mathbb{H}}.$$

We localize k, such that $k = \chi_{[0,\delta]}$, $\delta \ll 1$. We have

$$\begin{aligned}
\|K\|^2 &= \int_{\Gamma\backslash\mathbb{H}} K(z,\ w)^2\mathrm{d}\mu(z)\\
&= \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma,\gamma'} k(u(\gamma' z,\ w))k(u(\gamma\gamma', w))\mathrm{d}\mu(z)\\
&= \sum_\gamma \int_{\mathbb{H}} k(u(z,\ w))k(u(z,\ \gamma w))\mathrm{d}\mu(z). \qquad (0.1)
\end{aligned}$$

If $u(z,w) < \delta$ and $u(z,\gamma w) < \delta$, then we have

$$u(w,\gamma w) < 4\delta(\delta+1) \le 8\delta.$$

Then, by polar coordinates

$$\begin{aligned}
(0.1) &\ll \int_{\{z:\, u(z,w)<\delta\}} \sharp\{r:\ u(w,\gamma w) \le 8\delta\}\mathrm{d}\mu(z)\\
&= \mathrm{vol}\{z : u(z,w) < \delta\}\\
&= 4\pi\delta.
\end{aligned}$$

Lemma. *$\delta \ll h(t) \ll \delta$, if $t \ll \frac{1}{\sqrt{\delta}}$.*

Proof. We have

$$\begin{aligned} h(t) &= \int_{\mathbb{H}} k(u(z,\ i))y^s \mathrm{d}\mu(z) \\ &= \int_{B_\delta=\{z\ :\ u(z,i)<\delta\}} y^s \mathrm{d}\mu(z). \end{aligned}$$

And

$$u(z,i) < \delta \Rightarrow |y-1| < \sqrt{\delta} \Rightarrow |y^s - 1| \le \delta |y-1| \le \frac{1}{2},$$

if $t \ll \frac{1}{\sqrt{\delta}}$.

Thus,

$$\Big|\int_{B_\delta} y^s \mathrm{d}\mu(z) - \mathrm{vol}(B_\delta)\Big| \le \frac{1}{2}\mathrm{vol}(B_\delta)$$

Hence,

$$\frac{1}{2}\mathrm{vol}(B_\delta) \le h(t) \le 2\mathrm{vol}(B_\delta).$$

□

From Bessel's inequality, we get

$$\delta^2 \Big(\sum_{|t_j| \le \frac{1}{\sqrt{8\delta}}} |u_j(w)|^2) + \frac{1}{4\pi} \int_{-\frac{1}{\sqrt{8\delta}}}^{\frac{1}{\sqrt{8\delta}}} |E(w;\ \frac{1}{2} + ir)|^2 \mathrm{d}r \Big) \ll \delta.$$

If we take $\delta = \frac{1}{T^2}$ and $T \gg 1$, we get weak local Weyl law

$$\sum_{t_j < T} |u_j(z)|^2 + \frac{1}{4\pi} \int_{-T}^{T} |E(w;\ \frac{1}{2} + ir)|^2 \mathrm{d}r \ll T^2.$$

One can take it uniformly in z, $T^2 + T\mathrm{Im} z$ for $\mathrm{Im} z \gg 1$.

Remark. By integrating over z, one gets

$$\sharp\{t_j < T\} \ll t^2.$$

Corollary . *If k is any point-pair invariant function, K is the automorphic kernel,*

$$K(z,w) = \sum_j h(t_j) u_j(z) \overline{u_j(w)} + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(r) E(z;\ \frac{1}{2} + ir) \overline{E(w;\ \frac{1}{2} + ir)} \mathrm{d}r.$$

Suppose that

$$|h(t)| \le H(t), \quad t \in \mathbb{R},$$

where H is monotone, then

$$K(z,w) = \sum_{\frac{1}{2} < s_j \le 1} h(t_j) u_j(z) \overline{u_j(w)} + O\Big(\int_0^\infty (H(t)+1)t \mathrm{d}t \Big).$$

Proof. Use partial summation or integral.

Hyperbolic lattice point problem: We study

$$\sharp\{\gamma \in \Gamma \ : \ d(\gamma z, w) < R)\} \text{ as } R \to \infty,$$

where z, w are fixed. All alternatively,

$$\sharp\{\gamma \in \Gamma \ : \ u(\gamma z, w) < X)\} =: P(X).$$

(Recall that, in $\mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$, this is the Gauss circle problem, $\sharp\{(a,b) : \ a^2 + b^2 < R\} = \pi R + O(\sqrt{R})$),

$$\sharp\{(a,b) : \ a^2 - b^2 < R\} = \sum_{n<R} \tau(n) \sim \text{area of a truncated hyperbolic},$$

$$\sharp\{(a,b) : \ a^2 - b^2 = n\} \sim \tau(n),$$

where $\tau(n)$ is the number of divisors. Our goal is to get an error term for

$$P(X) - 4\pi X,$$

$$P(X) = K(z,w) \quad \text{for } k = \chi_{[0,X]}.$$

The main term should be the exceptional eigenvalue $\frac{1}{2} < s_j \leq 1$. For that, we bound for h.

Using k,

$$P(X) \leq K(z,w) \leq P(X+Y),$$

where Y is a parameter. Recall

$$q(v) = \int_v^\infty \frac{k(u)}{\sqrt{u-v}} \mathrm{d}u,$$

$$g(r) = q(e^r + e^{-r} - 2),$$

$$h(t) = \int_{-\infty}^{+\infty} e^{irt} g(r) \mathrm{d}r.$$

For our k, for any f, by partial integration we get

$$\int_0^\infty f(u)k(u)\mathrm{d}u = \frac{1}{Y}\int_X^{X+Y} F(t)\mathrm{d}t,$$

where $F(t) = \int_0^t f(x)\mathrm{d}x$. For

$$f(x) = \begin{cases} \frac{1}{\sqrt{x-v}}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} F(t) &= \min(0, \sqrt{t-v})q(v) = \frac{1}{Y}\int_X^{X+Y} F(t)\mathrm{d}t \\ &= \frac{1}{Y}\Big[\min(0,(X+Y-v)^{3/2}) - \min(0,(X-v)^{3/2})\Big], \end{aligned}$$

$$g'(r) = q'(e^r + e^{-r} - 2)(e^r - e^{-r}),$$

and

$$\begin{aligned} -q'(v) &= \frac{1}{Y}\Big[\min(0,(X+Y-v)-\min(0,(X-v)\Big] \\ &\leq q'(0) \ll \sqrt{Y}. \end{aligned}$$

We have

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} e^{irt} g(r)\mathrm{d}r \\ &= \frac{1}{t^2}\int_{-\infty}^{\infty} e^{irt} g''(r)\mathrm{d}r. \end{aligned}$$

Trivially, we get

$$|h(t)| \ll \frac{1}{t^2}\int_{-\infty}^{\infty} |g''(r)|\mathrm{d}r = \frac{1}{t^2}$$

$$\text{total variation of g'} \ll \frac{1}{t^2}\max|g'| \ll \frac{1}{Y}\frac{1}{t^2}X\sqrt{Y} = \frac{1}{t^2}\frac{X}{\sqrt{Y}}.$$

So $\sharp$ of extreme of $g' \ll 5$. If $\Gamma = SL_2(\mathbb{Z})$, $t \in \mathbb{R}$,

$$h(\frac{i}{2}) = \int_0^{\infty} k(u)\mathrm{d}u = 4\pi(X+Y).$$

$$P(X) = \frac{4\pi}{\mathrm{area}(\Gamma\backslash\mathbb{H})}X + O(Y + \frac{X}{\sqrt{Y}}),$$

$u_0 = \frac{1}{\sqrt{\mathrm{area}(\Gamma\backslash\mathbb{H})}}$. Consequently, if $Y = X^{2/3} = O(X^{2/3})$,

$$\sharp\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a^2+b^2+c^2+d^2 \leq X\right\} = 3X + O(X^{2/3}).$$

For

$$\Gamma = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) :\ a \equiv d\ (2),\ b \equiv c\ (2)\right\}$$

which is conjugate to $\Gamma_0(2)$, $r(n) = \sharp\left\{(a,b) :\ n = a^2+b^2\right\}$, we have

$$\sum_{n\leq X} r(n)r(n+1) = 4X + O(X^{2/3}).$$

We consider functions on $\Gamma\backslash\mathbb{H}$, where $\mathbb{H} = SL_2(\mathbb{R})/K$. We have $\Gamma\backslash\mathbb{H} = \Gamma\backslash SL_2(\mathbb{R})/K$ and have the following relations : functions on $\Gamma\backslash\mathbb{H}$ = functions on $\Gamma\backslash SL_2(\mathbb{R})/K$ = left Γ-invariant and right K-invariant functions on $G = SL_2(\mathbb{R})$

Look at

$$C(\Gamma\backslash G) = \{\text{left } \Gamma\text{-invariant functions on } G\}$$

G acts on $\Gamma\backslash\mathbb{H}$ by right translation. Therefore G acts on $C(\Gamma\backslash G)$. Take $g \in G,\ f \in C(\Gamma\backslash G)$

$$[R(g)f](x) = f(xg)$$

$$R(g): \ C(\Gamma\backslash G) \hookleftarrow$$

$R: \ G \to \mathrm{Aut}(C(\Gamma\backslash G))$ is homomorphism of groups, i. e. $(R, C(\Gamma\backslash G))$ is a representation of G. Consider $L^2(\Gamma\backslash G, dg)$, dg is Haar measure. $R: \ G \to \mathfrak{U}(L^2(\Gamma\backslash G))$, $\mathfrak{U}$ is the group of unitary operators.

Main problem: Decompose R into irreducible representation.

$$C(\Gamma\backslash G)^K = \{v \in C(\Gamma\backslash G) : R(k)_v = v, \forall k \in K\},$$

$$L^2(\Gamma\backslash\mathbb{H}) = (L^2(\Gamma\backslash G)^K.$$

One can classify irreducible unitary representation of G. For any irreducible representation (π, v) of G (i.e. no closed G-invariant subspaces of V, or equivalently, no non-scalar operators which commute with $\pi(g), \forall g \in G$).

$$V^K = \{v \in V; \pi(k)v = v, \forall k \in K\}$$

is at most one-dimensional.

In the case where Γ is uniform (co-compact).

$$L^2(\Gamma\backslash G) = \oplus\pi_i,$$

this is the direct sum of Irreducible representation. Therefore

$$L^2(\Gamma\backslash\mathbb{H}) = (L^2(\Gamma\backslash G)^K = \oplus\pi_i^K.$$

$(m(\pi) < \infty$, where $m(\pi) =$ multiplicity of π in $L^2(\Gamma\backslash G))$. Δ preserves the isotypic compact of π. Eigenfunction of Δ $\leftrightarrow$irreducible representation π which occurs in $L^2(\Gamma\backslash G)$ s. t. $\pi^K \neq 0$.

$$\dim\{\phi : (\Delta + \lambda)\phi = 0\} = m(\pi),$$

$$\lambda \leftrightarrow \pi_\lambda.$$

where $\lambda = \frac{1}{4}+s^2$, and $s \in \mathbb{R}\cup[-\frac{i}{2}, \frac{i}{2}]$. This is a parametrization of irreducible unitary representation, such that $\pi^K \neq 0$.

Selberg's conjecture:

$$\lambda_1(\Gamma(N)) \geq \frac{1}{4}, \ s_i \in \mathbb{R}, \ i > 0.$$

Representation with $\lambda \geq \frac{1}{4}$ (i. e. $s \in \mathbb{R}$) have decaying matrix coefficients. If (π, V) is an irreducible admissible representation, a matric coefficient of π is a function of the form

$$g \xrightarrow{\text{matrix coefficient}} (\pi(g)v_1, v_2),$$

where $v \in V$, and

$$g = k_1 \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} k_2.$$

$$\int_G |f(g)|^{2+\varepsilon} \mathrm{dg} < \infty, \quad \varepsilon > 0.$$

What about other representations which occur in $L^2(\Gamma\backslash G)$ modular forms + others.

Discrete series representations: for any $k = 1, 2, \ldots$ Denote σ_k is the matrix coefficient in $L^2(G)$, we have

$$L^2(G) = \oplus_{k=1}^{\infty} \sigma_k \oplus \int_{s\in i\mathbb{R}} \pi_{\frac{1}{4}+s^2} \mathrm{ds},$$

(c.f. $L^2(\mathbb{R}) = \oplus \int_{\mathbb{R}} e^{ix}\mathrm{dx}$)

Denote $m(\sigma_k)$ = dimension of modular forms of weight $k+1$, we have weight $1 \to$ "limit of discrete series".

For $\Gamma = SL_2(z)$, Hecke operators : $\{T_n\}_{n\in\mathbb{Z}}$. Recall strong approximation:

$$SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) \cdot SL_2(\mathbb{R}) \cdot \prod_{p<\infty} SL_2(\mathbb{Z}_p)$$

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}) \longleftrightarrow SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}) \cdot \prod_{p<\infty} K_p$$

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}) / \prod_{p\leq\infty} K_p \longleftrightarrow SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K_\infty = \Gamma(1)\backslash\mathbb{H}$$

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})/K_\infty \prod_{p<\infty} K_p(N) \longleftrightarrow \Gamma(N)\backslash SL_2\mathbb{R}/K_\infty = \Gamma(N)\backslash\mathbb{H}$$

where $K_p(N)$ is congruence subgroup of K_p. This is to say

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})/K_\infty = \varprojlim \Gamma(N)\backslash\mathbb{H}$$

$$C\left(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})\right)^{K_\infty} = \varinjlim L^2\left(\Gamma(N)\backslash\mathbb{H}\right)$$

The new problem: decomposition of $L^2(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$.
First part: what is the continuous spectrum? Answer: explicit construction using Eisenstein series.

What about discrete part? The analogue of Selberg conjecture in the principal becomes a bound on the Fourier transform of Maass forms.

Ramannujan-Petersson conjecture :

$$\|a_p^*\| \leq 2.$$

The bound was proved by Delign.
Modular forms: $a_p^* = \frac{a_p}{p^{\frac{k-1}{2}}}$, trivial bound:

$$a_p^* \leq \sqrt{p}.$$

Let

$$\lambda_1 \leq \frac{1}{4} + \delta^2 \ \ (s \leq \frac{1}{2} + \delta),$$

$$a_p \leq p^\delta + p^{-\delta}.$$

So for the best known δ is $\frac{7}{64}$.

e. g.

$$P(X) = \sum_{\frac{1}{2}\leq c_j<1} cX^{sj} + O(X^{\frac{2}{3}}).$$

Kim-Shahidi : For $\delta \leq \frac{1}{6}$,

$$P(X) = \frac{\pi}{\text{area}(\Gamma\backslash\mathbb{H})}X + O(X^{\frac{2}{3}}).$$

Higher weight case: $G(\mathbb{Q})\backslash G(\mathbb{A})$, G is linear reductive group.

What is the decomposition into irreducible representation? $\Gamma\backslash G/K$ ring of invariant differential operators. what is the continuous spectrum?(Langlands). what is the discrete part? There is an analogue of Ramanujan conjecture(By Arthur). Langlands functoriality suggests a relation between the automorphic spectrum of $L^2(G_i(\mathbb{Q})\backslash G_i(\mathbb{A}))$ for many pairs (G_i, G_j) of reductive groups.