2010

2. Let $f: \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points x in $\mathbb{R}$ where f is continuous is a countable intersection of open sets.

4. Suppose $f : [a, b] \to \mathbf{R}$ be a L^1-integrable function. Extend f to be 0 outside the interval $[a, b]$. Let

$$\phi(x) = \frac{1}{2h} \int_{x-h}^{x+h} f$$

Show that

$$\int_a^b |\phi| \leq \int_a^b |f|.$$

5. Suppose $f \in L^1[0, 2\pi]$, $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$, prove that

1) $\sum_{|n|=0}^{\infty} |\hat{f}(n)|^2 < \infty$ implies $f \in L^2[0, 2\pi]$,

2) $\sum_{n} |n\hat{f}(n)| < \infty$ implies that $f = f_0, a.e., f_0 \in C^1[0, 2\pi]$,
where $C^1[0, 2\pi]$ is the space of functions f over $[0, 1]$ such that both f and its derivative f' are continuous functions.

2011

4. Let $S = \{x \in \mathbb{R} \mid |x - \frac{p}{q}| \leq c/q^3, \text{for all } p, q \in \mathbb{Z}, q > 0, c > 0\}$, show that S is uncountable and its measure is zero.

5. Let $C([0, 1])$ denote the Banach space of real valued continuous functions on [0, 1] with the sup norm, and suppose that $X \subset C([0, 1])$ is a dense linear subspace. Suppose $l : X \to \mathbb{R}$ is a linear map (not assumed to be continuous in any sense) such that $l(f) \geq 0$ if $f \in X$ and $f \geq 0$. Show that there is a unique Borel measure μ on [0, 1] such that $l(f) = \int f d\mu$ for all $f \in X$.

2012

4. Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$ is a measurable function on $\mathbb{R}$.

3. In the unit interval [0, 1] consider a subset
$E = \{x|$ in the decimal expansion of x there is no $4\}$,
show that E is measurable and calculate its measure.

4. Let $1 < p < \infty, L^p([0,1], dm)$ be the completion of $C[0,1]$ with the norm: $||f||p = (\int_0^1 |f(x)|^p dm)^{\frac{1}{p}}$, where dm is the Lebesgue measure. Show that $\lim_{\lambda\to\infty} \lambda^p m(x||f(x)| > \lambda) = 0$.

2013

1. Suppose that f is an integrable function on $\mathbf{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x||f(x)| > \alpha\}$. Prove that:

$$\int_{\mathbf{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

2. Let f be a function of bounded variation on $[a, b]$, f_1 its generalized derivative as a measure, i.e. $f(x) - f(a) = \int_a^x f_1(y)dy$ for every $x \in [a, b]$ and $f_1(x)$ is an integrable function on $[a, b]$. Let f' be its weak derivative as a generalized function, i.e. $\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx$, for any smooth function $g(x)$ on $[a, b], g(a) = g(b) = 0$. Show that:

a) If f is absolutely continuous, then $f' = f_1$.

b) If the weak derivative f' of f is an integrable function on $[a, b]$, then $f(x)$ is equal to an absolutely continuous function outside a set of measure zero.

2014

4. Let $U(\xi)$ be a bounded function on $\mathbb{R}$ with finitely many points of discontinuity, prove that

$$P_U(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} U(\xi)d\xi$$

is a harmonic function on the upper half plane $\{z \in \mathbb{C}|Imz > 0\}$ and it converges to $U(\xi)$ as $z \to \xi$ at a point ξ where $U(\xi)$ is continuous.

5. Let $f \in L^2(\mathbb{R})$ and let $\hat{f}$ be its Fourier transform. Prove that

$$\int_{-\infty}^{\infty} x^2|f(x)|^2dx \int_{-\infty}^{\infty} \xi^2|\hat{f}(\xi)|^2d\xi \geq \frac{(\int_{-\infty}^{\infty}|f(x)|^2dx)^2}{16\pi^2},$$

under the condition that the two integrals on the left are bounded.

(Hint: Operators $f(x) \to xf(x)$ and $\hat{f}(\xi) \to \xi\hat{f}(\xi)$ after Fourier transform are non-commuting operators. The inequality is a version of the uncertainty principle.)

4. Let $D \subset \mathbb{R}^n$ be a bounded open set, $f : \bar{D} \to \bar{D}$ is a smooth map such that its Jacobian $\left|\frac{\partial f}{\partial x}\right| \equiv 1$, where $\bar{D}$ denotes the closure of D. Prove

(a) for each small ball $B_\epsilon(x)$, there exists a positive integer k such that $f^k(B_\epsilon(x)) \cap B_\epsilon(x) \neq \varnothing$, where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ;

(b) there exists $x \in \bar{D}$ and a sequence $k_1, k_2, \cdots k_j, \cdots$ such that $f^{k_j}(x) \to x$ as $k_j \to \infty$.

2015

1. Let $f_n \in L^2(R)$ be a sequence of measurable functions over the line, $f_n \to f$ almost everywhere. Let $||f_n||_{L^2} \to ||f||_{L^2}$, prove that $||f_n - f||_{L^2} \to 0$.

2. Let f be a continuous function on $[a, b]$, define $M_n = \int_a^b f(x)x^n dx$. Suppose that $M_n = 0$ for all n, show that $f(x) = 0$ for all x.

6. Let H_1 be the Sobolev space on the unit interval $[0, 1]$, i.e. the Hilbert space consisting of functions $f \in L^2([0, 1])$ such that

$$||f||_1^2 = \Sigma_{n=-\infty}^{\infty}(1 + n^2)|\hat{f}(n)|^2 < \infty;$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^1 f(x)e^{-2\pi i n x}dx$$

are Fourier coefficients of f. Show that there exists constant $C > 0$ such that

$$||f||_{L^\infty} \leq C||f||_1$$

for all $f \in H_1$, where $||.||_{L^\infty}$ stands for the usual supremum norm. (Hint: Use Fourier series.)

2. Let f be a Lebesgue integrable function over $[a, b + \delta], \delta > 0$, prove that

$$\lim_{h\to 0}\int_a^b |f(x + h) - f(x)|dx \to 0.$$

2016

2. Let $p > 0$ and suppose $f_n, f \in L^p[0, 1]$ and $||f_n - f||_p = (\int_0^1 |f_n(x) - f(x)|^p dx)^{\frac{1}{p}} \to 0$ as $n \to \infty$.

a) Show that for every $\epsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in [0, 1]||f_n(x) - f(x)| > \epsilon\}) = 0.$$

Here m is the Lebesgue measure.

b) Show that there exists a subsequence f_{n_j} such that $f_{n_j}(x) \to f(x)$ for almost every $x \in [0, 1]$.

1. Suppose that F is continuous on $[a,b], F^{'}(x)$ exists for every $x \in (a,b), F^{'}(x)$ is integrable. Prove that F is absolutely continuous and

$$F(b) - F(a) = \int_a^b F^{'}(x)dx.$$

2. Suppose that f is integrable on $\mathbf{R}^n$, let $K_\delta(x) = \delta^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{\delta}}$ for each $\delta > 0$. Prove that the convolution

$$(f * K_\delta)(x) = \int_{\mathbf{R}^n} f(x-y)K_\delta(y)dy$$

is integrable and $||(f * K_\delta) - f||_{L^1(\mathbf{R}^n)} \rightarrow 0$, as $\delta \rightarrow 0$.