Langlands picture of automorphic forms and L-functions

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§4 The global functional equation (Mar. 13)

Let F be a global field. Suppose that c is a quasi-character of $F^\times\backslash A_F^\times$ with factorization $c=\chi|\cdot|^s$, where χ is unitary character and $\sigma=\mathrm{Re}(s)$ is the exponent of c. For $f\in S(\mathbb{A}_F)$, i.e. $f=\prod_v f_v$ and $f_v=1_{O_v}$ for almost all v, define the global zeta function as

$$\begin{aligned}\zeta(f,c) &= \int_{\mathbb{A}_F^\times} f(x)c(x)\,d^\times x\\ &= \prod_v \int_{F_v^\times} f_v(x_v)c_v(x_v)d^\times x_v\end{aligned}$$

which is convergent for $\sigma=\mathrm{Re}(s)>1$. Indeed,

$$\prod_{\substack{v\\ unramified}} \int_{F_v^\times} |f_v(x_v)c_v(x_v)|\,d^\times x_v = \prod_v{}' \int_{O_v} |x_v|^{\mathrm{Re}(s)}d^\times x_v = \prod_v{}' \frac{1}{1-q_v^{-\sigma}}$$

converges for $\sigma>1$.

Theorem 4.1 (Global Theorem, Tate). *$\zeta(f,c)$ extends to a meromorphic function of s and satisfies the functional equation*

$$\zeta(f,c)=\zeta(\widehat{f},c^\vee).$$

It is holomorphic everywhere except when $c\neq|\cdot|^{-i\tau}$, $\tau\in\mathbb{R}$ in which case it has simple poles at $s=i\tau$ and $s=1+i\tau$ with residues given by

$$-\mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1)f(0)\quad\text{and}\quad\mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1)\widehat{f}(0)$$

respectively.

In order to prove the theorem, we need the following Riemann-Roch theorem, which is also called Poisson summation formula.

Lemma 4.2 (Riemann-Roch). *Let x be an idele of K and let $f\in S(\mathbb{A}_F)$. Then*

$$\sum_{\gamma\in F} f(\gamma x)=\frac{1}{|x|}\sum_{\gamma\in F}\widehat{f}(\gamma x^{-1})$$

Proof : Since $\mathbb{A}_F^\times=(0,\infty)\mathbb{A}_F^1$, we have

$$\begin{aligned}\zeta(f,c) &= \int_{\mathbb{A}_F^\times} f(x)c(x)\,d^\times x\\ &= \int_0^\infty \int_{\mathbb{A}_F^1} f(tx)c(tx)d^\times x\frac{dt}{t}.\end{aligned}$$

Denote

$$\zeta_t(f,c) = \int_{\mathbb{A}_F^1} f(tx)c(tx)d^\times x.$$

Claim:

$$\zeta_t(f,c) = \zeta_{t^{-1}}(\widehat{f}, c^\vee) + \widehat{f}(0)\int_{F^\times\backslash\mathbb{A}_F^1} c^\vee(\frac{x}{t})d^\times x - f(0)\int_{F^\times\backslash\mathbb{A}_F^1} c(xt)d^\times x. \tag{4.1}$$

In fact, we have

$$\zeta_t(f,c) = \int_{F^\times\backslash\mathbb{A}_F^1} \sum_{\gamma\in F^\times} f(\gamma tx)c(\gamma tx)d^\times x.$$

So then

$$\begin{aligned}
\zeta_t(f,c) + f(0)\int_{F^\times\backslash\mathbb{A}_F^1} c(xt)d^\times x &= \int_{F^\times\backslash\mathbb{A}_F^1} c(tx)\sum_{\gamma\in F} f(\gamma tx)d^\times x \\
\textit{Riemann} - \textit{Roch} \longrightarrow \qquad &= \int_{F^\times\backslash\mathbb{A}_F^1} \frac{c(tx)}{|tx|}\sum_{\gamma\in F} \widehat{f}(\gamma t^{-1}x^{-1})d^\times x \\
\text{replace } x \text{ with } x^{-1} \longrightarrow \qquad &= \widehat{f}(0)\int_{F^\times\backslash\mathbb{A}_F^1} c^\vee(\frac{x}{t})d^\times x + \int_{F^\times\backslash\mathbb{A}_F^1} c^\vee(\frac{x}{t})\sum_{\gamma\in F^\times}\widehat{f}(\gamma\frac{x}{t})d^\times x \\
&= \widehat{f}(0)\int_{F^\times\backslash\mathbb{A}_F^1} c^\vee(\frac{x}{t})d^\times x + \zeta_{t^{-1}}(\widehat{f}, c^\vee)
\end{aligned}$$

which yields the claim.

Now back to proof of the theorem. We may write

$$\zeta(f,c) = \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(t,c)\frac{dt}{t}.$$

The second integral is simply

$$\int\limits_{\{x\in\mathbb{A}_F^\times : |x|\geq 1\}} f(x)c(x)d^\times x$$

which converges normally for all s. Indeed, the convergence is better for small σ, and since we know it converges for $\sigma > 1$, it converges everywhere.

For the first integral, we have

$$\int_0^1 \zeta_t(f,c)\frac{dt}{t} = \int_0^1 \zeta_{t^{-1}}(\widehat{f}, c^\vee)\frac{dt}{t} + E,$$

where

$$E = \int_0^1 \left[\widehat{f}(0)\int_{F^\times\backslash\mathbb{A}_F^1} c^\vee(\frac{x}{t})d^\times x - f(0)\int_{F^\times\backslash\mathbb{A}_F^1} c(xt)d^\times x\right]\frac{dt}{t}.$$

Via substitution of t^{-1} for t, we find that

$$\int_0^1 \zeta_{t^{-1}}(\widehat{f}, c^\vee)\frac{dt}{t} = \int_1^\infty \zeta_t(\widehat{f}, c^\vee)\frac{dt}{t},$$

which is also convergent for all s by the argument above.

It remains to analyze E. We distinguish the following two cases.

Case 1: $c = \chi|\cdot|^s$ is nontrivial on $F^\times\backslash\mathbb{A}_F^1$. By the orthogonality of characters on compact group, we have

$$\int_{F^\times\backslash\mathbb{A}_F^1} c(tx)d^\times x = 0,$$

which implies that $E = 0$.

Case 2: $c = \chi|\cdot|^s$ is trivial on $F^\times\backslash\mathbb{A}_F^1$. We know that in fact $c = |\cdot|^{s'}$, where $s' = s - i\tau$, for some real τ, and in this case

$$\int_{F^\times\backslash\mathbb{A}_F^1} c(tx)d^\times x = |t|^{s'}\mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1).$$

Thus

$$\begin{aligned} E &= \int_0^1 \left[\widehat{f}(0)t^{s'-1}\mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1) - f(0)t^{s'}\mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1)\right]\frac{dt}{t} \\ &= \mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1)\left[\frac{\widehat{f}(0)}{s'-1} - \frac{f(0)}{s'}\right], \end{aligned}$$

which has poles at $s' = 0, 1$.

So far, we have

$$\begin{aligned} \zeta(f,c) &= \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\widehat{f},c^\vee)\frac{dt}{t} + E(f,c) \\ &= \int_1^\infty \int_{\mathbb{A}_F^\times} f(tx)c(tx)d^\times x\frac{dt}{t} + \int_1^\infty \int_{\mathbb{A}_F^\times} \widehat{f}(tx)c^\vee(tx)d^\times x\frac{dt}{t} + E(f,c) \end{aligned}$$

Since

$$\widehat{\widehat{f}} = f(-x) \quad \text{and} \quad ((c)^\vee)^\vee = c,$$

it follows also that

$$\begin{aligned} \zeta(\widehat{f},c^\vee) &= \int_1^\infty \zeta_t(\widehat{f},c^\vee)\frac{dt}{t} + \int_1^\infty \zeta_t(\widehat{\widehat{f}},c^\vee)\frac{dt}{t} + E(\widehat{f},c^\vee) \\ &= \int_1^\infty \int_{\mathbb{A}_F^\times} \widehat{f}(tx)c^\vee(tx)d^\times x\frac{dt}{t} + \int_1^\infty \int_{\mathbb{A}_F^\times} f(-tx)c(tx)d^\times x\frac{dt}{t} + E(\widehat{f},c^\vee). \end{aligned}$$

So we only need to show

$$E(f,c) = E(\widehat{f},c^\vee).$$

But

$$E(f,c) = \mathrm{Vol}(F^\times\backslash\mathbb{A}_F^1)\left[\frac{s'\widehat{f}(0) - (s'-1)f(0))}{s'(1-s')}\right] = E(\widehat{f},c^\vee).$$

Now the functional equation follows by the above arguments. $\square$

Theorem 4.3 (Tate, Hecke)**.** *Let χ be any unitary characters of $F^\times\backslash\mathbb{A}_F^\times$, and recall $L(s,\chi) = L(\chi|\cdot|^s) = \prod_{p\leq\infty} L(s,\chi_v)$. Then $L(s,\chi)$ extends meromophically to all s in $\mathbb{C}$ and satisfies the functional equation*

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\overline{\chi})$$

where $\varepsilon(s,\chi)$ is non-zero.

Proof. Assume $L(s,\chi)$ meromorphic. Tate's main theorem says

$$\zeta(s,f,\chi)=\zeta(1-s,\widehat{f},\overline{\chi})=\prod_v \zeta(1-s,\widehat{f}_v,\overline{\chi_v})$$

Divide by $\prod_v \zeta(s,f_v,\chi_v)$ to get

$$1=\prod_v \frac{\zeta(s,\widehat{f}_v,\overline{\chi_v})}{\zeta(s,f_v,\chi_v)}=\prod_v \frac{\varepsilon(s,\chi_v)L(1-s,\overline{\chi_v})}{L(s,\chi_v)}=\frac{\varepsilon(s,\chi)L(1-s,\overline{\chi})}{L(s,\chi)},$$

where

$$\varepsilon(s,\chi)=\prod_v \varepsilon(s,\chi_v)$$

□

Remark 1. Hecke looked at the function

$$L_S(s,\chi)=\prod_{\substack{v\\ \text{unramified}}} L(s,\chi_v)$$

and had a functional equation between $L_S(s,\chi)$ and $L_S(1-s,\overline{\chi})$, i.e., he considered the L-function constructed only from finite unramified primes.

Remark 2. Taking $F=Q$, and $\chi=1$, we get the functional equation of Riemann's zeta function

$$\xi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\xi(1-s).$$

Remark 3. We give some modern interpretations here. Following Tate, Langlands tries to give the functional equation of $L(s,\pi)$ where $\pi=\bigotimes\pi_v$ is an automorphic representation of a reductive group, and generalizes the theory of $GL(1)$. Sarnak follows Hecke and considers the L-functions over almost all places.

Automorphic form on $GL(1)$. Assume F is a global field. An automorphic form on $GL_1(F)$ is just a Grossencharacter(a unitary character of idele of F module $F^\times$). In the next two weeks, we will study automorphic forms on $GL(2)$.