Graduate Algebra Final Exam, Spring 2011
School of Mathematics, Shandong University

Instructions: *This is a closed book, closed notes exam! Work problems 1-5. Then do one problem from 6-8, making a total of six problems. If you have extra time, you can answer the optional bonus problem. Show all details in your proof in English. You have two and a half hours to complete this test. Good luck!*

14:30-17:00; June 29, 2011

1. **(10 points)** Let a and b belong to the group G. If $ab = ba$ and $|a| = m$ $|b| = n$, where $(m, n) = 1$. Show that $|ab| = mn$ and $\langle a \rangle \cap \langle b \rangle = 1$.

2. **(20 points)** (1), State the orbit-stabilizer theorem and the class equation.
(2), Show that for prime p, every group of order p^2 is abelian.

3. **(20 points)** (1), State Schur's lemma.
(2), Let G be an abelian group. Then any irreducible representation ρ of G over finite-dimensional complex vector space V has degree one.

4. **(20 points)** (1), Show that Gauss domain $\mathbb{Z}[i]$ is a Euclidean domain.
(2), Find the gcd and the lcm of $11 + 3i$ and $8 - 3i$ in $\mathbb{Z}[i]$.

5. **(20 points)** (1), Find a splitting field K of $f(x) = x^3 - 2$ over $\mathbb{Q}$ and $[K : \mathbb{Q}]$.
(2), Find the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
(3), Show that $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_3$.

------------------------- ******** -------------------------

6. **(10 points)** Let $\pi = (1, 2, 3)(1, 2, 5, 6) \in S_6$. Find $|\pi|$, and then find π^{2009}.

7. **(10 points)** Is 2 an irreducible or prime element in the ring $\mathbb{Z}[2i]$?

8. **(10 points)** Find all $a, b \in \mathbb{Z}_3$ for which $\mathbb{Z}_3[x]/\langle x^2 + ax + b \rangle$ is a field. How many elements of $\mathbb{Z}_3[x]/\langle x^2 + ax + b \rangle$ are there when it's a field?

------------------------- ******** -------------------------

9. **(Bonus Problem)** The alternating group A_5 is simple.

Guanghua Ji
School of Mathematics
Shandong University
Jinan, Shandong 250100