Langlands picture of automorphic forms and L-functions

— Lecture series at Shandong University, Mar. 2009

STEPHEN GELBART

§3 Tate's Local theory (Mar. 10)

The local L-functions are defined as follows.

If F is non-Archimedean, we set

$$L(c) = L(s,\chi) = \begin{cases} \frac{1}{1-\chi(\varpi)q^{-s}} & \text{if } \chi \text{ is unramified;} \\ 1 & \text{otherwise,} \end{cases}$$

where ϖ is the uniformizer parameter and $|\varpi| = q^{-1}$.

If $F = \mathbb{R}$, then $U_F = \{\pm 1\}$. Set

$$L(c) = L(s,\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) & \text{if } \chi = 1; \\ \Gamma_{\mathbb{R}}(s+1) & \text{if } \chi = \text{sgn}. \end{cases}$$

If $F = \mathbb{C}$, then $U_F = S^1$, and χ takes the form $\chi_n : e^{i\theta} \mapsto e^{in\theta}$, for some $n \in \mathbb{Z}$. Set

$$L(c) = L(s,\chi_n) = \Gamma_{\mathbb{C}}\left(s + \frac{|n|}{2}\right) = (2\pi)^{-\left(s+\frac{|n|}{2}\right)}\Gamma\left(s + \frac{|n|}{2}\right).$$

Theorem 3.1 (Tate's local theorem). *Let $f \in \mathcal{S}(F)$ and $c(x) = |x|^s\chi(x)$ with χ unitary of exponent $\sigma = \Re(s)$. Take*

$$\zeta(f,c) := \zeta(f,\chi,s) = \int_{F^\times} f(x)c(x)\, d^\times x$$

and let $c^\vee = c^{-1}|\cdot|$. Then the following statements hold:

(A) *$\zeta(f,c)$ is absolutely convergent for $\sigma > 0$.*

(B) *If $0 < \sigma < 1$, there is a functional equation*

$$\zeta(\widehat{f}, c^\vee) = \gamma(c,\psi,\mathrm{d}x)\zeta(f,c) \tag{3.1}$$

for some $\gamma(c,\psi,\mathrm{d}x)$ independent of f, which in fact is meromorphic as a function of s.

(C) *For all $s \in \mathbb{C}$, there is a non-zero factor $\varepsilon(c,\psi,\mathrm{d}x)$ which satisfies the relation*

$$\gamma(c,\psi,\mathrm{d}x) = \varepsilon(c,\psi,\mathrm{d}x)\frac{L(c^\vee)}{L(c)}. \tag{3.2}$$

Remark. The global L-function is defined as the product of local L-functions over $p \le \infty$. In the next section, we will see that the product of $\gamma(c,\psi,\mathrm{d}x)$ over $p \le \infty$ becomes 1, and so we get the functional equation of the global L-function.

Proof of **(A)**. Let

$$I(f,c) = \int_{F^\times} |f(x)||x|^\sigma \mathrm{d}^\times x.$$

It is sufficient to show that $I(f,c)$ is convergent for $\sigma > 0$. This is obvious for F Archimedean, since $f \in \mathcal{S}(F)$ implies that f is smooth and rapidly decreasing. For F non-Archimedean,

$f \in \mathcal{S}(F)$, i.e. f is locally constant and compact supported. We know $\{\mathfrak{p}^r\}_{r=0}^{\infty}$ forms a basis of compact neighborhoods of $0 \in F$. Thus it suffices to consider the special case $f = 1_{\mathfrak{p}^r}$. We have

$$\begin{aligned} I(f,c) &= \int_{\mathfrak{p}^r-\{0\}} |x|^{\sigma} \mathrm{d}^{\times}x \\ &= \sum_{j=r}^{\infty} \int_{A_j=\varpi^j U_F} |x|^{\sigma} \mathrm{d}^{\times}x \\ &= \sum_{j=r}^{\infty} q^{-j\sigma} \int_{U_F} \mathrm{d}^{\times}x \\ &= \frac{q^{-r\sigma}}{1-q^{-\sigma}} \mathrm{Vol}(U_F, \mathrm{d}^{\times}x). \end{aligned}$$

Obviously $I(f,c)$ converges for $\sigma > 0$. $\square$

In order to prove **(B)**, we need the following lemma

Lemma 3.2. *Let $\sigma = \Re s$. For any $f, g \in \mathcal{S}(F)$ and $0 < \sigma < 1$, we have*

$$\zeta(f,c)\zeta(\widehat{g},c^{\vee}) = \zeta(\widehat{f},c^{\vee})\zeta(g,c). \tag{3.3}$$

Proof.

$$\begin{aligned} \zeta(f,c)\zeta(\widehat{g},c^{\vee}) &= \int_{F^{\times}} \int_{F^{\times}} f(x)\widehat{g}(y)c(xy^{-1})|y| \mathrm{d}^{\times}x \mathrm{d}^{\times}y \\ &= \int_{F^{\times}} \int_{F^{\times}} f(x)\widehat{g}(xy)c(y^{-1})|xy| \mathrm{d}^{\times}x \mathrm{d}^{\times}y \\ &= \int_{F^{\times}} \{f,g\}(y)c(y^{-1})|y| \mathrm{d}^{\times}y, \end{aligned}$$

where

$$\{f,g\}(y) = \int_{F^{\times}} f(x)\widehat{g}(xy)|x| \mathrm{d}^{\times}x.$$

Claim. $\{f,g\} = \{g,f\}$.

Indeed, since $|x|\mathrm{d}^{\times}x = c \cdot \mathrm{d}x$,

$$\begin{aligned} \{f,g\}(y) &= \int_{F^{\times}} f(x)\widehat{g}(xy)|x| \mathrm{d}^{\times}x \\ &= \int_{F^{\times}} \int_{F} f(x)g(z)\psi(xyz)|x| \mathrm{d}z \mathrm{d}^{\times}x \\ &= c\int_{F\times F} f(x)g(z)\psi(xyz) \mathrm{d}z \mathrm{d}x \\ &= \{g,f\}(y). \end{aligned}$$

This establishes the claim and the lemma follows. $\square$

Proof of **(B)**. Fix a function $f_0 \in \mathcal{S}(F)$ and put

$$\gamma(c, \psi, \mathrm{d}x) = \frac{\zeta(\widehat{f_0}, c^\vee)}{\zeta(f_0, c)}.$$

Then by the preceding lemma, γ is independent of the choice of f_0, and we have

$$\zeta(\widehat{f}, c^\vee) = \gamma(c, \psi, \mathrm{d}x)\zeta(f, c)$$

as asserted. Since $\zeta(f, c)$ is defined for c with exponent $\sigma = \Re s > 0$, while $\zeta(\widehat{f}, c^\vee)$ is defined for $\sigma < 1$, we will get the requisite meromorphic continuation of $\zeta(f, c)$ if we can show that $\gamma(c, \psi, \mathrm{d}x)$ is meromorphic everywhere. This will follow from the proof of **(C)**, where we will in fact compute $\gamma(c, \psi, \mathrm{d}x)$ for a suitable f_0. □

Proof of **(C)**. **Case** $F = \mathbb{R}$.

Take $\mathrm{d}x$ to be the usual Lebesgue measure and choose $\psi(x) = e^{-2\pi i x}$. We distinguish two cases.

If $c(x) = |x|^s$, then we take $f(x) = e^{-\pi x^2}$ which is clearly in $\mathcal{S}(\mathbb{R})$. We compute

$$\begin{aligned} \zeta(f, c) &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \mathrm{d}^\times x \\ &= 2\int_0^\infty e^{-\pi x^2} x^{s-1} \mathrm{d}x \\ &= \pi^{-\frac{s}{2}} \int_0^\infty e^{-u} u^{\frac{s}{2}-1} \mathrm{d}u \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right). \end{aligned}$$

By the definition of $L(c)$, we know that $\zeta(f, c) = L(c)$ for all characters in this case. On the other hand, we have

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y} \mathrm{d}x = f(y).$$

Thus

$$\begin{aligned} \zeta(\widehat{f}, c^\vee) &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{1-s} \mathrm{d}^\times x \\ &= \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) = L(c^\vee). \end{aligned}$$

Therefore, for $c(x) = |x|^s$, we have

$$\gamma(c, \psi, \mathrm{d}x) = \frac{L(c^\vee)}{L(c)}$$

and $\varepsilon(c, \psi, \mathrm{d}x) = 1$.

If $c(x) = |x|^s \mathrm{sgn}(x)$, then we take $f(x) = xe^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$. Since $\mathrm{sgn}(x) = x/|x|$, we find that

$$\begin{aligned}\zeta(f, c) &= \int_{\mathbb{R}^\times} xe^{-\pi x^2} \cdot \frac{x}{|x|} \cdot |x|^s \mathrm{d}^\times x \\ &= 2\int_0^\infty e^{-\pi x^2} x^s \mathrm{d}x \\ &= \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right).\end{aligned}$$

Thus $\zeta(f, c) = L(c)$ by definition. Through integration by contour, we have

$$\widehat{f}(y) = iye^{-\pi y^2}.$$

Thus

$$\begin{aligned}\zeta(\widehat{f}, c^\vee) &= i\int_{\mathbb{R}^\times} xe^{-\pi x^2} \cdot \frac{x}{|x|} \cdot |x|^{1-s} \mathrm{d}^\times x \\ &= iL(c^\vee).\end{aligned}$$

Therefore, for $c(x) = |x|^s \mathrm{sgn}(x)$, we have

$$\gamma(c, \psi, \mathrm{d}x) = i\frac{L(c^\vee)}{L(c)}$$

and $\varepsilon(c, \psi, \mathrm{d}x) = i$.

Case $F = \mathbb{C}$.

For $z = x + iy$, we have $|z| = z\overline{z} = x^2 + y^2$, $\mathrm{d}z = 2\mathrm{d}x\mathrm{d}y$, and

$$\mathrm{d}^\times z = \frac{2\mathrm{d}x\mathrm{d}y}{x^2 + y^2} = 2\frac{\mathrm{d}r\mathrm{d}\theta}{r}.$$

The additive character is given by

$$\psi(z) = e^{-2\pi i z\overline{z}}.$$

Since $F^\times = \mathbb{C}^* = \mathbb{R}_+^\times \times S^1$, every character of $\mathbb{C}^*$ takes the form

$$c_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

for some uniquely defined complex s and integral n. Put

$$f_n(z) = \begin{cases} (2\pi)^{-1}\overline{z}^n e^{-2\pi z\overline{z}} & \text{for } n \leq 0 \\ (2\pi)^{-1} z^{-n} e^{-2\pi z\overline{z}} & \text{for } n < 0. \end{cases}$$

We give the following results as exercise.

$$\gamma(c_{s,n}, \psi, \mathrm{d}z) = i^{|n|}\frac{L(c_{s,n}^\vee)}{L(c_{s,n})},$$

and

$$\varepsilon(c_{s,n}, \psi, \mathrm{d}z) = i^{|n|}.$$

Case F is non-Archimedean.

We call $\mathfrak{p}^m$ is the conductor of a non trivial additive character ψ, if $m = \inf\{r \in \mathbb{Z}, \psi|_{\mathfrak{p}^m} \equiv 1\}$, where $\mathfrak{p}^0 = O_F$. We call $\mathfrak{p}^n$ the conductor of a multiplicative character $c : F^\times \mapsto \mathbb{C}^*$, if $U_n = 1 + \mathfrak{p}^n (n \leq 0)$ is the largest subgroup on which c is 1. c is unramified if its conductor is $U_0 = U_F = O_F^\times$.

Assuming c is unramified, i.e. $c(x) = |x|^s$ for $x \in O_F - \{0\}$. Let ψ be an additive character with conductor $\mathfrak{p}^m$. Define

$$f(x) = \begin{cases} \psi(x) & \text{if } x \in \mathfrak{p}^m \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \zeta(f, c) &= \int_{\mathfrak{p}^m - \{0\}} \psi(x)|x|^s \mathrm{d}^\times x \\ &= \sum_{j=m}^{\infty} \int_{\varpi^j U_F} |x|^s \mathrm{d}^\times x \\ &= \sum_{j=m}^{\infty} q^{-js} \int_{U_F} \mathrm{d}^\times x \\ &= q^{-ms} \mathrm{Vol}(U_F, \mathrm{d}^\times x) L(c_{s,0}), \end{aligned}$$

where

$$L(c_{s,0}) = L(s, 1) = \frac{1}{1 - q^{-s}}.$$

Exercise. $\widehat{f}(y) = \mathrm{Vol}(\mathfrak{p}^{\mathfrak{m}}, \mathrm{d}x) 1_{O_F}(y)$.

According to above , we have

$$\zeta(\widehat{f}, c^\vee) = \int_{F-0} \widehat{f}(y) c^\vee(y) \mathrm{d}^\times y \tag{3.4}$$

$$= \mathrm{Vol}(\mathfrak{p}^m, \mathrm{d}x) \int_{O_F - 0} c^\vee(y) \mathrm{d}^\times y \tag{3.5}$$

$$= \mathrm{Vol}(\mathfrak{p}^m, \mathrm{d}x) \sum_{k \geq 0} q^{-k(1-s)} \int_{U_F} c^\vee(y) \mathrm{d}^\times y \tag{3.6}$$

$$= \mathrm{Vol}(\mathfrak{p}^m, \mathrm{d}x) \mathrm{Vol}(U_F, \mathrm{d}^\times x) L(c^\vee). \tag{3.7}$$

Therefore, we get

$$\gamma(c, \psi, \mathrm{d}x) = q^{ms} \mathrm{Vol}(\mathfrak{p}^m, \mathrm{d}x) \frac{L(c^\vee)}{L(c)},$$

Clearly $\gamma(c, \psi, \mathrm{d}x)$ is meromorphic and $\varepsilon(c, \psi, \mathrm{d}x)$ is nonzero and

$$\varepsilon(c, \psi, \mathrm{d}x) = q^{ms} \mathrm{Vol}(\mathfrak{p}^m, \mathrm{d}x).$$

Assuming c is ramified, one can take $c(x) = |x|^s \omega(\frac{x}{|x|})$, where ω has ramified conductor $\mathfrak{p}^n$ and

$$f(x) = \begin{cases} \psi(x) & \text{if } x \in \mathfrak{p}^{m-n}; \\ 0 & \text{otherwise.} \end{cases}$$

Let ψ be an additive conductor $\mathfrak{p}^m$ and . We state result here and leave caculation to readers.

$$\begin{aligned} \zeta(f, c) &= q^{-(m-n)} g(\omega, \psi_{\varpi^{m-n}}), \\ \zeta(\widehat{f}, c^\vee) &= \mathrm{Vol}(\mathfrak{p}^{m-n}, \mathrm{d}x) \mathrm{Vol}(1 + \mathfrak{p}^n, \mathrm{d}^\times x) \omega(-1). \end{aligned}$$

where

$$g(\omega, \psi_{\varpi^{m-n}}) = \int_{U_F} \omega(u)\psi_{\varpi^{m-n}}(u)\mathrm{d}^\times x$$

is the Gauss sum. Since $L(c^\vee) = L(c) = 1$, we get

$$\gamma(c, \psi, \mathrm{d}x) = \varepsilon(\chi, \psi, \mathrm{d}x) = q^{(m-n)s} \frac{\mathrm{Vol}(\mathfrak{p}^{m-n}, \mathrm{d}x)\mathrm{Vol}(1 + \mathfrak{p}^n, \mathrm{d}^\times x)\omega(-1)}{g(\omega, \psi_{\varpi^{m-n}})}.$$

Again $\varepsilon(\chi, \psi, \mathrm{d}x)$ is nonzero and equals $\gamma(c, \psi, \mathrm{d}x)$, it is easy to see that $\gamma(c, \psi, \mathrm{d}x)$ is meromorphic as a function of s. $\square$