S.-T. Yau College Student Mathematics Contests

2010

1.

a) Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ $(z \in D)$. Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}. \qquad (z \in D)$$

b) For any finite complex value a, prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{i\theta}| d\theta = \max\{\log |a|, 0\}.$$

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y-axis, the function has limit 1, while when approaching any point in the negative half of the y-axis, the function has limit -1.

6. Suppose $\Omega \subset \mathbf{R}^3$ to be a simply connected domain and $\Omega_1 \subset \Omega$ with boundary Γ. Let u be a harmonic function in Ω and $M_0 = (x_0, y_0, z_0) \in \Omega_1$. Calculate the integral:

$$II = -\int\int_\Gamma (u\frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r}\frac{\partial u}{\partial n})dS,$$

where $\frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2 + (y-x_0)^2 + (z-x_0)^2}}$ and $\frac{\partial}{\partial n}$ denotes the out normal derivative with respect to boundary Γ of the domain Ω_1. (Hint: use the formula $\frac{\partial v}{\partial n}dS = \frac{\partial v}{\partial x}dy \wedge dz + \frac{\partial v}{\partial y}dz \wedge dx + \frac{\partial v}{\partial z}dx \wedge dy$.)

$k=1$

b) From

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin(x^2)dx$.

2011

1. a) Compute the integral: $\int_{-\infty}^{\infty} \frac{x \cos x dx}{(x^2+1)(x^2+2)}$,

b) Show that there is a continuous function $f : [0, +\infty) \to (-\infty, +\infty)$ such that $f \not\equiv 0$ and $f(4x) = f(2x) + f(x)$.

3. Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \setminus (-\infty, -1]$ to the unit disc.

2012

2. Let V be a simply connected region in the complex plane and $V \neq \mathbb{C}$. Let a, b be two distinct points in V. Let ϕ_1, ϕ_2 be two one-to-one holomorphic maps of V onto itself. If $\phi_1(a) = \phi_2(a)$ and $\phi_1(b) = \phi_2(b)$, show that $\phi_1(z) = \phi_2(z)$ for all $z \in V$.

1. Compute the integral

$$\int_0^{\infty} \frac{x^p}{1+x^2} dx, -1 < p < 1.$$

2. Construct a one to one conformal mapping from the region

$$U = \{z \in \mathbb{C} | |z - \frac{i}{2}| < \frac{1}{2}\} / \{z | |z - \frac{i}{4}| < \frac{1}{4}\}$$

onto the unit disk.

2013

1. Suppose $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ is the open unit disk in the complex plane. Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \qquad (1)$$

for all z in Δ. If equality holds in (1) for some $z_0 \in \Delta$, show that $f \in \text{Aut}(\Delta)$, and that

$$\frac{|f'(z)|}{1 - |f(z)|^2} = \frac{1}{1 - |z|^2}$$

for all $z \in \Delta$.

4. Let u be a positive harmonic function over the punctured complex plane $\mathbf{C}/\{0\}$. Show that u must be a constant function.

2014

1. Calculate the integral:

$$\int_0^\infty \frac{\log x}{1+x^2}dx.$$

3. Prove that any bounded analytic function F over $\{z|r<|z|<R\}$ can be written as $F(z)=z^\alpha f(z)$, where f is an analytic function over the disk $\{z||z|<R\}$ and α is a constant.

5. Let u be a subharmonic function over a domain $\Omega \subset \mathbf{C}$, i.e., it is twice differentiable and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Prove that u achieves its maximum in the interior of Ω only when u is a constant.

2. Let $f_1, ..., f_n$ are analytic functions on $D=\{z||z|<1\}$ and continuous on $\bar{D}$, prove that $\phi(z)=|f_1(z)|+|f_2(z)|+...+|f_n(z)|$ achieves maximum values at the boundary ∂D.

3. Prove that if there is a conformal mapping between the annulus $\{z|r_1<|z|<r_2\}$ and the annulus $\{z|\rho_1<|z|<\rho_2\}$, then $\frac{r_2}{r_1}=\frac{\rho_2}{\rho_1}$.

4. Let $U(\xi)$ be a bounded function on $\mathbb{R}$ with finitely many points of discontinuity, prove that

$$P_U(x)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{(x-\xi)^2+y^2}U(\xi)d\xi$$

is a harmonic function on the upper half plane $\{z\in\mathbb{C}|Imz>0\}$ and it converges to $U(\xi)$ as $z\to\xi$ at a point ξ where $U(\xi)$ is continuous.

6. Let Ω be an open domain in the complex plane $\mathbb{C}$. Let $\mathbb{H}$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω.

a) Show that $\mathbb{H}$ is a closed subspace of $L^2(\Omega)$, and hence is a Hilbert space with inner product

$$(f,g)=\int_\Omega f(z)\bar{g}(z)dxdy, \text{ where } z=x+iy.$$

b) If $\{\phi_n\}_{n=0}^\infty$ is an orthonormal basis of $\mathbb{H}$, then

$$\Sigma_{n=0}^{\infty}|\phi_n(z)|^2 \leq \frac{c^2}{d(z,\Omega^c)}, \text{ for } z \in \Omega.$$

c) The sum

$$B(z,w) = \Sigma_{n=0}^{\infty}\phi_n(z)\bar{\phi}_n(w)$$

converges absolutely for $(z,w) \in \Omega \times \Omega$, and is independent of the choice of the orthonormal basis.

2015

4. Let $f : U \to U$ be a holomorphic function with U a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f'(0) = 1$, prove that $f(z) = z$.

3. Determine all entire functions f that satisfying the inequality

$$|f(z)| \leq |z|^2|Im(z)|^2$$

for z sufficiently large.

4. Describe all holomorphic functions over the unit disk $D = \{z||z| \leq 1\}$ which maps the boundary of the disk into the boundary of the disk.