Langlands picture of automorphic forms and L-functions

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Stephen Gelbart

§5 Automorphic Forms(Mar. 17)

Let F be a global number field, $\mathbb{A}$ the adele ring of F and ψ an automorphic form on GL_1. Denote $G = GL_n$, $Z =$ center of G. We consider the following space.

$$\begin{aligned} L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}, \psi) = \{\varphi : G_F\backslash G_{\mathbb{A}} \to \mathbb{C} \text{ measurable};\\ \varphi(zg) = \psi(z)\varphi(g), \forall z \in Z_{\mathbb{A}};\\ \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} |\varphi(g)|^2 \mathrm{d}^*g < \infty\}, \end{aligned}$$

where d^*g is the respective Haar measure on $G_{\mathbb{A}}$.

Let $R^\psi(g)$ be the right regular unitary representation of $G_{\mathbb{A}}$ on $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}, \psi)$. The representation decomposes as follows:

$$R^\psi(g) = \oplus_j \pi_j + \int \pi_s \mathrm{d}s,$$

where π_j and π_s are irreducible unitary representation of $G_{\mathbb{A}}$, which are in the discrete part and continuous part, respectively.

An irreducible unitary representation π of $G_{\mathbb{A}}$ is called *automorphic*, if it is realized as some π_j or π_s(for some G). π is called *cuspidal* if it is not one-dimensional and realizable as some π_j.

Remark. For $G = GL_1$, we have $Z_{\mathbb{A}} = G_{\mathbb{A}}$. Thus

$$L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}, \psi) = \{\psi\}.$$

This shows that automorphic forms on GL_1 are just Grossencharacters.

Every automorphic representation π has an L-series associated to it. Langlands conjectured that every L-series that arose in number theory has to appear in the above list of automorphic forms. This kind of transfer of L-series is called *functoriality.*

Example 1. Class field theory

Let L/F be an abelian extension of number fields, σ an irreducible representation of $G(L/F)$, and $L(s, \sigma)$ the corresponding Artin L-series. Via Artin's reciprocity law, one has

$$L(s, \sigma) = L(s, \chi_\sigma),$$

where χ_σ is the corresponding Grossencharacter of F, i.e. automorphic form of GL_1 over F, and $L(s, \chi_\sigma)$ is the L-series attached to χ_σ which was studied by Hecke and Tate. This implies that the Artin-L series are just L-series of automorphic forms of GL_1.

Example 2. Non-abelian Artin L-series

Similarly, Let L/F be a Galois extension of number fields, and σ an irreducible representation of $G(L/F)$ with dimension n, $L(s, \sigma)$ the corresponding Artin L-series. Via Langlands functoriality, it is conjectured that

$$L(s, \sigma) = L(s, \pi_\sigma),$$

where π_σ is an automorphic form on GL_n over F. In fact, it is pretty known in the case $n = 2$.

Example 3. Symmetric L-series

Assume π is an automorphic form of GL_2. It is conjectured that

$$L(s, \mathrm{sym}^m(\pi)) = L(s, \Pi_{\mathrm{sym^m}}),$$

where $\Pi_{\mathrm{sym^m}}$ is an $m+1$ dimensional automorhpic form. This conjecture implies the Ramanujan conjecture and Sato-Tate conjecture.

Example 4. Hasse-Weil L-series

Assume E is an elliptic curve, and $L(s, E)$ the corresponding Hasse-Weil L-series. It was partly proved by A.Wiles in his famous paper that

$$L(s, E) = L(s, \pi_E),$$

where π_E is an automorphic form of GL_2.

This short sketch of the Langlands Program shows us the real importance of the case GL_2 and automorphic forms on it.

1. Classical Definition of Automorphic Forms

Going back to 1940's, we give the classical defination of automorphic forms. For simplicity, we only consider the case $\Gamma = SL(2, \mathbb{Z})$.

Definition 1 (Classical)**.** Let $f(z)$ be a complex valued function on $\mathbb{H}$. Consider the following conditions.

(1) f is holomorphic;

(2) f is "modular" or "automorphic", i.e.

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma;$$

(3) f is "holomorphic at ∞";

(4) f is "cuspidal at ∞".

We call f *modular form* or *automorphic form*, if f satisfies the first three conditions. Moreover, we call f *cusp form*, if f satisfies all the conditions above.

Remark. It follows from condition (2) that $f(z+1) = f(z)$, thus we have

$$f(z) = \sum_n a_n \mathrm{e}^{2\pi i n z}.$$

f "holomorphic at ∞" means only non-negative n appear; it is cuspidal at ∞ means only positive n appear.

Denote $M_k(\Gamma)$ the space of modular forms and $S_k(\Gamma)$ the space of cusp forms. We have $S_k(\Gamma) \subset M_k(\Gamma)$ and $\dim S_k(\Gamma)$ is finite. The Petersson inner product on $S_k(\Gamma)$ is defined as follows:

$$(f, g) = \iint_{\mathbb{H}} f(z)\bar{g}(z) y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Example 1. The most famous example is the Ramanujan function,

$$\Delta(z) = \mathrm{e}^{\pi \mathrm{i}z}(1-\mathrm{e}^{2\pi \mathrm{i}z})^{24} = \sum_{n=1}^{\infty} \tau(n)\mathrm{e}^{2\pi \mathrm{i}z}.$$

It is a modular form of weight 12 for $\Gamma = SL_2(\mathbb{Z})$ and generates the one-dimensional space $S_{12}(\Gamma)$. Ramanujan conjectured that

$$\tau(p)\tau(q) = \tau(pq) \quad \text{for } (p,q)=1,$$
$$\tau(n) = O(n^{11/2+\varepsilon}).$$

These have been proved by Mordell and Deligne respectively.

The Hecke operators $T(p) : S_k(\Gamma) \to S_k(\Gamma)$ are defined by

$$T(p)f(z) = p^{k-1} \sum_{\substack{a>0 \\ ad=p}} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) d^{-k}.$$

One can show $T(p)$ are Hermitian with respect to the Petersson inner product, and commute with each other. So there exists an orthornormal basis $\{f_j\}$ for $S_k(\Gamma)$ such that

$$T(p)f_j = \lambda_p f_j.$$

Therefore, if $f \in S_k(\Gamma)$ is an eigenfunction of T_p for all p and satisfies $a(1)=1$, we have

$$\lambda_p = a(p),$$
$$a(p)a(q) = a(pq) \quad \text{for } (p,q)=1.$$

The L-series attached to $f \in S_k(\Gamma)$ is defined as follows

$$L(s,f) = (2\pi)^{-s}\Gamma(s)D(s,f),$$

where

$$D(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is a Dirichlet series. Hecke studied $L(s,f)$ and gave the following theorem.

Theorem 5.1 (Hecke, around 1935). *We have*

(1) *$D(s,f)$ converges in some half-plane. $L(s,\chi)$ is analytically continuous for all s in $\mathbb{C}$ and satisfies*

$$L(s,f) = \varepsilon(s,f)L(k-s,f),$$

where

$$varepsilon(s,f) = \mathrm{i}^k.$$

Moreover, the converse is also true.

(2) *$D(s,f)$ will be Eulerian if and only if*

$$T(p)f = a_p f \quad \textit{for any } p.$$

In this case,

$$D(s,f) = \prod_{p<\infty} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

Remark. Hecke Operator is very much like the Grossencharacter.

2. Group theory in Automorphic forms

We start from 1951 when Gelfond firstly looked at $\mathbb{H}$ as $K\backslash SL_2(\mathbb{R})$. Let $G = SL(2,\mathbb{R})$. We know G acts on $\mathbb{H}$ transitively, and

$$\mathrm{stab}(i) = K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}.$$

Therefore we have $\mathbb{H} = K\backslash G$. Iwasawa decomposition implies that $G = NAK$, where

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}.$$

Denoting $B = NA$, we have the bijective $\mathbb{H} \to B$ given by

$$x + \mathrm{i}y \mapsto \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}$$

Therefore, one can use coordinates (x, y, θ) to determine g. For any $g \in G$, we have

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5.1}$$

With respect to (5.1), the Haar measure d^*g on G can be expressed as

$$\int_G \varphi(g)\mathrm{d}^*g = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \varphi(x, y, \theta)\mathrm{d}x\frac{\mathrm{d}y}{y}\mathrm{d}\theta.$$

Theorem 5.2. *Denote $j(g, \mathrm{i}) = (c\mathrm{i} + d)$. For $f \in S_k(\Gamma)$,*

$$f \to \varphi(g) = \varphi_f(g) = f(g\mathrm{i})j(g, \mathrm{i})^{-k}$$

gives an isomorphism from $S_k(\Gamma)$ to $A_k(\Gamma)$, the space of functions satisfying

(1) $\varphi(\gamma g) = \varphi(g), \forall\gamma \in \Gamma$.

(2) $\varphi(g\kappa(\theta)) = \varphi(g)\mathrm{e}^{-\mathrm{i}k\theta}, \forall\kappa(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(3) *φ is bounded, in particular, $\varphi \in L^2(\Gamma\backslash G)$.*

(4) *φ is cuspidal on G, i.e.*

$$\int_0^1 \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\mathrm{d}x = 0,\ \forall\, g \in G.$$

(5) $\Delta\varphi = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\varphi$.

Sketch of Proof: It is easy to check that $j(\gamma g, \mathrm{i}) = j(\gamma, g\mathrm{i})j(g, i)$. Thus

$$\begin{aligned} \varphi(\gamma g) &= j(\gamma g, \mathrm{i})^{-k} f(\gamma g i) \\ &= j(\gamma g, \mathrm{i})^{-k} j(\gamma, g i)^k f(g\mathrm{i}) \\ &= j(g, \mathrm{i})^{-k} f(g\mathrm{i}) \\ &= \varphi(g). \end{aligned}$$

Similarly, one can show φ_f satisfies conditions (2),(3),(4) and the map is injective and isometry. In fact, f being holomorphic implies that φ_f is "nice" for Δ. Thus the injectivity and condition (5) follow by next theorem. $\square$

Theorem 5.3. *Let $R(g)$ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Then Δ operates on smooth functions of $L^2(\Gamma\backslash G)$ in terms of the coordinates (x, y, θ) as*

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta}.$$

One has $R\Delta = \Delta R$. Via Schur's lemma, Δ acts on any G-invariant irreducible subspace of $L^2(\Gamma\backslash G)$ as scalar.

Remark. Other functions in $L^(\Gamma\backslash G)$ are corresponding to maass wave forms!

Definition 2 (General Definition)**.** An automorphic cuspidal form φ on G is a smooth complex valued function such that

(1) $\varphi(\gamma g) = \varphi(g)$, $\forall\, \gamma \in \Gamma$;
(2) φ is right K-finite;
(3) φ is cuspidal on G, i.e.

$$\int_0^1 \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = 0, \quad \forall\, g \in G;$$

(4) φ is bounded;
(5) φ is an eigenfunction of Δ.

Example. Maass wave cusp forms

Let $W_s(\Gamma)$ denote the space of smooth functions on $\Gamma\backslash G$, such that φ is K-invariant, bounded, cuspidal and $\Delta\varphi = \left(\frac{1-s^2}{4}\right)\varphi$. Then $f(z) \mapsto \varphi(g) = f(g\mathrm{i})$ gives an isomorphism from $W_s(\Gamma)$ to the space of functions in $\mathbb{H}$ which is cuspidal and satisfies

$$\Delta^* f = \left(\frac{1-s^2}{4}\right) f,$$

where

$$\Delta^* = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

They are the functions that Maass studied.

In the next section, we will use eigenvalues and Δ-eigenspaces to classify the irreducible cuspidal sub-representations of $R(g)$.