Final Exam of Real Analysis

School of Mathematics, Shandong University

Instructions: *This is a closed book, closed notes exam! Show all details in your proof in English. You have two hours to complete this test. Good luck!*

注意事项：卷面分5分,试题总分95分. 其中卷面整洁,书写规范(5分);卷面较整洁,书写较规范(3分); 书写潦草，乱涂乱画(0分).

1. **(10 points) Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.**

2. **(20 points) (1) State Egorov's theorem and Lusin's theorem.**
(2) Prove Lusin's theorem.

3. **(10 points) (1) Let $E = \{(x, y) \in [0, 1]^2$: at least one of x, y is rational$\}$. Compute mE.**
(2) Compute $\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n} dx$.

4. **(15 points) Prove that**
(1) if $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, then $\lim_{t\to 0} \int_{\mathbb{R}^n} |f(x + t) - f(x)|^p dx = 0$. (Hint: $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.)
(2) if $f \in L^1(\mathbb{R}^n)$ and $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ixy}dx$, then $\lim_{|y|\to\infty} \widehat{f}(y) = 0$

5. **(20 points) Let E be a measurable subset of $\mathbb{R}^n$ and $f, f_n \in L^2(E)$. Show that when $n \to \infty$, $\|f_n - f\|_2 \to 0$ if and only if $\|f_n\|_2 \to \|f\|_2$ and $\langle f_n, g\rangle \to \langle f, g\rangle$ for any $g \in L^2(E)$.**

6. **(20 points) Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies the Lipschitz condition $|f(x) - f(y)| \le M|x - y|$ for some $M > 0$ and all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'(x)| \le M$ for a.e. x.**