On Some Topics in Automorphic Representations

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Introduction

Automorphic Representations

Automorphic L-functions

Langlands Functoriality

Beyond the Genericity

Final Remarks
Acknowledgement

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Basic Structures of Numbers

► **Theorem (Fundamental Theorem of Arithmetic)**

For any \( r \in \mathbb{Q} \), there is prime numbers \( p_1, p_2, \cdots, p_t \) and integers \( e_1, e_2, \cdots, e_t \) such that

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r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.
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- It is a **multiplicative structure** in terms of primes.
- The additive structure in terms of primes should be the **Goldboch Conjecture**, which asserts the expression of even integers as sum of two primes, and is a much harder problem.
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- It is much easier for kids to learn addition of numbers than the multiplication of numbers.
- However, it seems that the multiplication has much better structure. The local-Global principle in modern number theory is one of the good examples related to the multiplicative structure of numbers.
- From $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, to know $r$ is equivalent to know all $p_i^{e_i}$, individually.
- To measure $r$ we use the usual absolute value; and to measure $p_i^{e_i}$ we use the so called p-adic absolute value.
p-adic Absolute Value

Given a prime $p$, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$. 
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For $r \in \mathbb{Q}^\times$, we have $\prod_v |r|_v = 1$. 
Locally Compact Topological Fields

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- They are only locally compact topological fields containing $\mathbb{Q}$ as a dense set.
- For $v = \infty$ or $p$, denote the Haar measure $dx_v$ on $\mathbb{Q}_v$, which is unique up to a constant.
- The Harmonic Analysis on $(\mathbb{Q}_v, dx_v)$ is expected to have deep impact in Number Theory.
The Riemann Zeta Function

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- The pole at \( s = 1 \) of \( \zeta(s) \) implies there are infinitely many primes!
- The \( p \)-factor has something to do with harmonic analysis over \( \mathbb{Q}_p \).
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- $\mathbb{A}$ is a locally compact ring containing all $\mathbb{Q}_v$; and $\mathbb{Q}$ is discrete in $\mathbb{A}$ such that $\mathbb{A}/\mathbb{Q}$ is compact.
- $(\mathbb{A}, \mathbb{Q})$ is a modern analogy of the classical pair $(\mathbb{R}, \mathbb{Z})$. 
For each $v$, there exists a Schwartz function $\phi_v$, such that:

$$\int_{\mathbb{Q} \times \mathbb{A}} \phi_v(x)|x|_v^s d^\times x_v = \begin{cases} 1 & \text{if } v = p, \\ \frac{1-p^{-s}}{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)} & \text{if } v = \infty. \end{cases}$$
Tate’s Thesis

- For each \( v \), \( \exists \) a Schwartz function \( \phi_v \), s.t.
  \[
  \int_{\mathbb{Q}^\times_v} \phi_v(x)|x|_v^s \, dx_v = \begin{cases} 
    1 & \text{if } v = p, \\
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  \end{cases}
  \]

- \( \exists \) a Schwartz function \( \phi = \otimes_v \phi_v \) on \( \mathbb{A} \), s.t.
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  \int_{\mathbb{A}^\times} \phi(x)|x|_\mathbb{A}^s \, dx = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \cdot \prod_p \frac{1}{1-p^{-s}}.
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Tate’s Thesis

- For each $\nu$, $\exists$ a Schwartz function $\phi_\nu$, s.t.

$$\int_{\mathbb{Q}_\nu^\times} \phi_\nu(x) |x|_\nu^s d^\times x_\nu = \begin{cases} 
\frac{1}{1-p^{-s}} & \text{if } \nu = p, \\
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\end{cases}$$

- $\exists$ a Schwartz function $\phi = \bigotimes_\nu \phi_\nu$ on $\mathbb{A}$, s.t.

$$\int_{\mathbb{A}^\times} \phi(x) |x|_\mathbb{A}^s d^\times x = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \cdot \prod_p \frac{1}{1 - p^{-s}}.$$ 

- The local-global relation in harmonic analysis approaches the local-global relation in arithmetic!
Modern Theory of Automorphic Forms

- Generalization from $GL(1)$ to general reductive algebraic groups defined over $\mathbb{Q}$. 
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- Generalization from $\zeta(s)$ to general automorphic L-functions.
- The Langlands Programme is to figure out the deep impacts of these generalizations to Number Theory and Arithmetic.
Algebraic Groups

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- For simplicity, we take $G = GL_n$, $SO_m$, $Sp_{2n}$, classical groups.
- For example, $SO_m = \{ g \in GL_m \mid ^t g J_m g = J_m, \det g = 1 \}$, with $J_m$ defined inductively by

$$J_m := \begin{pmatrix} 1 \\ J_{m-2} \\ 1 \end{pmatrix}.$$
Automorphic Functions

- $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$. 

$\mathbb{G}$ is a $G(\mathbb{A})$-module by $g \cdot f(x) := f(xg)$. 

\begin{itemize}
  \item $L^2(G)$ denotes the space of square-integrable functions:
  \end{itemize}

\[
\phi: \mathbb{Z}G(\mathbb{A}) \to \mathbb{C}
\] such that

\[
\int_{\mathbb{Z}G(\mathbb{A})} |\phi(g)|^2 \, dg < \infty.
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An automorphic function $\phi$ is called \textbf{cuspidal} if

$$\int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi(ng) dn = 0$$

for almost all $g \in G(\mathbb{A})$, where $N$ runs over the unipotent radical of all parabolic subgroups of $G$. 
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- \( L^2_c(G) \) denotes the subspace of \( L^2(G) \) generated by all irreducible cuspidal automorphic representations, which is called the **cuspidal spectrum** of \( G(\mathbb{A}) \).
Cuspidal Spectrum

- Theorem (Gelfand and Piatetski-Shapiro)

\[ L^2_c(G) = \bigoplus_{\pi \in G(\mathbb{A})^\vee} m_c(\pi) V_\pi \]

with \( m_c(\pi) < \infty \).
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For classical groups, \( G = SO_m \) or \( Sp_{2n} \), the Arthur conjecture asserts that

\[ m_c(\pi) \leq \begin{cases} 
1, & \text{if } G = SO_{2n+1}, Sp_{2n} \\
2, & \text{if } G = SO_{2n}.
\end{cases} \]
Known Cases of Cuspidal Multiplicity: \( m_c(\pi) \)

- \( G = GL_n, \ m_c(\pi) \leq 1 \) (J. Shalika; Piatetski-Shapiro)
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- $G = GSp_4$, $m_c(\pi) \leq 1$ with $\pi$ generic (D.-H. Jiang and D. Soudry)
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- $\pi_p$ is unramified if $\pi_p$ has nonzero $K_p = G(\mathbb{Z}_p)$-fixed vectors.
The Satake Theory of spherical functions

- \( \dim_{\mathbb{C}} V_{\pi_v}^{K_v} \leq 1 \), where

\[
V_{\pi_v}^{K_v} = \{ u \in V_{\pi_v} \mid \pi_v(h)(u) = u, \text{ for all } h \in K_v \}.
\]
The Satake Theory of spherical functions

- \( \dim_{\mathbb{C}} V_{\pi_v}^{K_v} \leq 1 \), where
  \[ V_{\pi_v}^{K_v} = \{ u \in V_{\pi_v} : \pi_v(h)(u) = u, \text{ for all } h \in K_v \}. \]

- Irreducible unramified representations of \( G(\mathbb{Q}_v) \) are parametrized by semi-simple conjugacy classes \( c(\pi_v) \) in the Langlands dual group \( L^G \), which is called the Satake parameter attached to \( \pi_v \).
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- Irreducible unramified representations of $G(\mathbb{Q}_v)$ are realized as the unramified irreducible constituent of the induced representation
  $$\text{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v),$$
  with unramified character $\chi_v$ of $T(\mathbb{Q}_v)$, where $B = TU$ is the Borel subgroup of $G$. 

Dihua Jiang University of Minnesota
On Some Topics in Automorphic Representations
The Langlands Dual Group of $G$

$(G, B, T)$ determines the root datum $(X, \Delta; X^\vee, \Delta^\vee)$. 

$\text{GL}_n(C) = \text{GL}_n(C)$ and $\text{SO}_{2n+1}(C) = \text{Sp}_{2n}(C)$. 

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Near-Equivalence Classes

- $S$ denotes any finite set of primes $p$ and $\infty$. 
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- Denote by $\mathcal{C}(G)$ the equivalence classes of all such sets $c(S)$.
- Denote by $\mathcal{A}(G)$ the set of irreducible cuspidal automorphic representations of $G(\mathbb{A})$ up to equivalence.
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For $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$, there exists $S_\pi$ such that for $p \notin S_\pi$, $\pi_p$ is unramified. Define $c(\pi) := c(S_\pi)$.
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$\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ are of **near-equivalence** if for almost all primes $p$, $\pi_p$ and $\pi'_p$ are equivalent.
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- **Problems:**
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**Problems:**

1. Describe the image \( c(\mathcal{A}(G)) \) in \( \mathcal{C}(G) \).
2. Describe the fibre \( \Pi_{c(\pi)} \).
3. Determine the structures of \( \pi \) in terms of \( c(\pi) \).
Rigidity of Cuspidal Automorphic Representations

- Theorem (Jacquet-Shalika, 1981)

For $G = GL_n$, $\prod_{c(\pi)}$ contains one member. (For $\pi, \pi'$ in $A(G)$, if $c(\pi) = c(\pi')$, then $\pi, \pi'$ are equivalent.)
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For $G = \text{SO}_{2n+1}$, $\Pi_{c(\pi)}$ contains at most one generic member; and if $\pi$ is tempered, $\Pi_{c(\pi)}$ contains at least one generic member.
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- A slight modification holds for general reductive groups. For classical groups, it is my on-going joint work with D. Soudry.
Tensor Product L-functions

- For $\pi \in \mathcal{A}(G)$ and $\tau \in \mathcal{A}(GL_m)$, define $S := S_{\pi,\tau}$, s.t. for $p \notin S$, both $\pi_p$ and $\tau_p$ are unramified.
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$$L^S(s, \pi \times \tau) := \prod_{p \not\in S} \frac{1}{\det(I - c(\pi_p) \otimes c(\tau_p)p^{-s})}.$$
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This is closely related to the structures of $c(\pi)$ and $\pi$, i.e. the local-global relations.
Weak Langlands Transfer Conjecture: Let $G$ and $H$ be $k$-split reductive algebraic groups and let $\rho$ be any group homomorphism

$$\rho : H^\vee(\mathbb{C}) \rightarrow G^\vee(\mathbb{C}).$$

For any $\sigma \in \mathcal{A}(H)$, $\exists$ a $\pi \in \mathcal{A}(G)$ (may not be cuspidal!) s.t.

$$c(\rho(\sigma)) = c(\pi)$$

as conjugacy classes in $G^\vee(\mathbb{C})$, where $c(\rho(\sigma)) = \{\rho(c(\sigma_v))\}$. 
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**The strong Langlands Functorial Transfer** requires compatibility at all local places or can be stated in terms of the complete tensor product $L$-functions.
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Refined Properties of Langlands Transfers

- Local-Global Compatibility:

\[ \text{Jiang-Soudry (2003): } \text{SO}_{2n+1} \Rightarrow \text{GL}_{2n} \]  
With explicit local descent, we obtain the local Langlands reciprocity map for $\text{SO}_{2n+1}$.

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The local descent in these cases are the work in progress of Jiang-Soudry, which also implies the existence of the local Langlands reciprocity map.

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- **Image of the Langlands Transfers:**
  
  - Ginzburg-Rallis-Soudry automorphic descent from $GL$ to classical groups characterizes the image of the Langlands transfer from classical groups to $GL$ (a series of papers in 1997-2005).
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Endoscopy and Poles of Certain L-functions

Theorem (Jiang 2006)

Let $\pi \in \mathcal{A}(SO_{2n+1})$ be cuspidal and generic.

1. The 2nd fundamental L-function $L(s, \pi, \omega_2)$ is holomorphic for $\Re(s) \geq \frac{1}{2}$ with possible pole at $s = 1$

2. The order of the pole at $s=1$ of $L(s, \pi, \omega_2)$ is $r - 1$ if and only if $\exists$ a partition $n = \sum_{i=1}^{r} n_i$ s.t. $\pi$ is an endoscopy transfer from the elliptic endoscopy group $SO_{2n_1+1} \times \cdots \times SO_{2n_r+1}$. 
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It is the work in progress of Ginzburg-Jiang to characterize the endoscopy transfers in terms of period of $\pi$, which will generalize our preliminary work in this aspect in 2001.
The Generalized Ramanujan Conjecture

- **GRC**: Any irreducible cuspidal automorphic representation is tempered

  - R. Howe and Piatetski-Shapiro (1977): GRC is not true for $G \neq \text{GL}$.
  - One of the refinements (Jiang, 2007): Any irreducible cuspidal automorphic representation with one local generic component is tempered.
  - This formulation holds for all known examples and is compatible with the Arthur conjecture on the discrete automorphic spectrum in general.
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- This formulation holds for all known examples and is compatible with the Arthur conjecture on the discrete automorphic spectrum in general.
The CAP Conjecture

Assume that \( G \) is \( \mathbb{Q} \)-quasisplit reductive group and \( G' \) be a \( \mathbb{Q} \)-inner form of \( G \). For any irreducible cuspidal automorphic representation \( \pi' \) of \( G' (\mathbb{A}) \), there exist a standard parabolic subgroup \( P = MN \) of \( G \), an irreducible generic unitary cuspidal automorphic representation \( \sigma \) of \( M (\mathbb{A}) \), and an unramified character \( \chi \) of \( M (\mathbb{A})^1 \setminus M (\mathbb{A}) \), such that \( \pi' \) is nearly equivalent to an irreducible constituent of the unitarily induced representation

\[
\text{Ind}^{G (\mathbb{A})}_{P (\mathbb{A})} (\sigma \otimes \chi).
\]
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- If $P$ is proper parabolic in $G$, $\pi'$ is called a CAP representation of $G'$. 
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- Jiang-Soudry (2007): For $G = SO_{2n+1}$, the CAP datum $(M, \sigma, \chi)$ is determined by $\pi'$, which is generalization of the rigidity of cuspidal automorphic representations.
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- For other classical groups, suitable modifications are needed, which is the work in progress of Jiang-Soudry.
The CAP Conjecture

▶ Jacquet-Shalika (1981): the CAP conjecture holds for $GL_n$. 
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- Gelbart-Rogawski-Soudry (1997): it holds for $U(3)$. 

Many families of CAP representations have been constructed, but we omit the details here.
The CAP Conjecture

- A. Badulescu (2007): it holds for $GL_m(D)$, where $D$ is a division algebra.
- Jiang-Soudry (2007): it holds for cuspidal automorphic representations of $SO_{2n+1}$ with special Bessel models.
- Gelbart-Rogawski-Soudry (1997): it holds for $U(3)$.
- Many families of CAP representations have been constructed, but we omit the details here.

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On Some Topics in Automorphic Representations
Final Remarks

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- The rational combination of the Arthur trace formula with the L-function and the Converse Theorem methods is definitely a very interesting approach for the near future.